Jacobson pairs and Bott-Duffin decompositions in rings

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ABSTRACT. We define the Bott-Duffin decompositions of elements in a ring, which generalize the strongly clean decompositions, and prove that the Bott-Duffin decompositions of $1 - ab$ are in a natural bijection with those of $1 - ba$. This bijection respects a number of well known additive decompositions of elements in a ring. For instance, the result implies that $1 - ab$ is strongly clean (respectively, strongly nil-clean, Drazin invertible, quasipolar, or pseudopolar) if and only if so is $1 - ba$. Examples and further applications are given.

Introduction

Given a ring R and two elements $a, b \in R$, we say that $(1 - ab, 1 - ba)$ is a *Jacobson pair*. The reason for this terminology is the following result, attributed to Jacobson.

Jacobson's Lemma. *If $\alpha = 1 - ab$ is a unit with inverse $s \in R$, then $\beta = 1 - ba$ is a unit with inverse $t = 1 + bsa \in R$.*

The equation $t = 1 + bsa$ is sometimes referred to as the "Desert Island Formula", in tribute to a vintage remark of Kaplansky duly quoted in the introductory section of [**21**]. There are many generalizations of this magic lemma. For instance, if (α, β) is a Jacobson pair and α is Drazin invertible, then β is also Drazin invertible [**7, 24**]. We say accordingly that *Jacobson's Lemma holds* for Drazin invertible elements. In [**3**] a formula for the Drazin inverse of β was found in terms of the Drazin inverse of α, and in [**21**] further expressions for the Drazin inverse were proven. Each of these formulas can be thought of as generalizations of the Desert Island Formula.

To give another example, again suppose (α, β) is a Jacobson pair. If α is (von Neumann) regular, then so is β; i.e. Jacobson's Lemma holds for regular elements. Furthermore, if s is an inner inverse for the element α, then $t = 1 + bsa$ is an inner inverse for β. It turns out that while the usual Desert Island Formula works, nevertheless it can still be improved and this was done by the authors in the recent paper [**20**].

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This paper is devoted to generalizing Jacobson's Lemma to a very large class of examples. We recapture the fact that Jacobson's Lemma holds for Drazin invertible elements, but extend this to (uniquely) strongly clean elements and strongly nil-clean elements, quasipolar and pseudopolar elements, among other examples introduced in §1. This is accomplished using the notion of a Bott-Duffin decomposition, named for the pioneering work done in [**2**], and to be formally defined in §2. The main theorem of this paper is Theorem 2.5, where we prove a natural bijection between the Bott-Duffin decompositions of any Jacobson pair. The remainder of the paper is devoted to exploring what other properties this natural one-to-one correspondence respects, and multiple examples are given which demonstrate the great generality that the Bott-Duffin decompositions provide.

As usual, given a ring R we write $\mathrm{U}(R)$ for the group of units, $\mathrm{idem}(R)$ for the set of idempotents, $J(R)$ for the Jacobson radical, and $\mathrm{nil}(R)$ for the set of nilpotent elements.

1. Generalized ABAB-decompositions

In this section we give a quick introduction to the notions of strongly clean, polar, quasipolar, and pseudopolar elements of rings. Many of these ring-theoretic properties were originally introduced in the context of linear algebra and matrix factorizations. Thus, in this section we emphasize a module-theoretic interpretation for certain additive decompositions of elements in rings.

This interpretation generalizes an observation which had its birth in the work of Nicholson [**23**] and further expansions by Diesl in [**8**] and [**9**, Lemma 2.8]. This connects certain additive element-wise decompositions of an endomorphism with corresponding module direct sum decompositions. Throughout, M_k will denote a right k-module, where k is some ring, and we will write endomorphisms on the left of M. (The ring k will not play an important role in what follows, and may be safely ignored. We also note in passing that every ring R can be viewed as the endomorphism ring $\mathrm{End}\,(R_R)$.) Given an endomorphism $\varphi \in \mathrm{End}\,(M_k)$ and a submodule $N \subseteq M$, we write $\varphi\!\restriction_N$ to denote the restriction of φ to the submodule N. In what follows, we will always restrict maps to φ-invariant submodules; that is $\varphi(N) \subseteq N$.

For any ring R and any subset $S \subset R$, the multiplicative *centralizer* (or commutant) of S in R is

$$C(S) := \{r \in R \,:\, rs = sr \text{ for all } s \in S\}.$$

We write $C^2(S)$ for the *double-centralizer* $C(C(S))$, and when $S = \{s\}$ is a single element, we write $C(s)$ rather than $C(\{s\})$.

PROPOSITION 1.1. *Fix an element $\alpha \in R = \mathrm{End}(M_k)$. Given $x \in R$, there exists an additive decomposition $\alpha = e + x$, where $e \in \mathrm{idem}(R) \cap C(\alpha)$, if and only if there exists a direct sum decomposition diagram*

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & \Big\downarrow {\scriptstyle \alpha\restriction_A = x\restriction_A} & & \Big\downarrow {\scriptstyle (1-\alpha)\restriction_B = -x\restriction_B} \\ M & = & A & \oplus & B. \end{array}$$

PROOF. ($\Rightarrow$): Assume $\alpha = e + x$ for some idempotent $e^2 = e \in R$ which commutes with α. Since $x = \alpha - e$, we see that e commutes with x as well. We find that $\alpha(1-e) = (e+x)(1-e) = x(1-e)$ and so $\alpha \restriction_{(1-e)M} = x \restriction_{(1-e)M}$. Similarly, we find that $(1-\alpha)e = (1-e-x)e = -xe$ so $(1-\alpha) \restriction_{eM} = -x \restriction_{eM}$. Setting $A := (1-e)M$ and $B := eM$, we have that these are α-invariant direct sum complements, which yields the needed module decomposition.

($\Leftarrow$): Suppose we are given a diagram as in the statement of the proposition. Let $e \in R$ be the (idempotent) projection to B with kernel A. Since both A and B are α-invariant, we see that e commutes with α. One checks directly that α and $e+x$ behave the same on both A and B, proving the equality $\alpha = e+x$. $\square$

This proposition motivates the following definition.

Definition 1.2 (**ABAB-Decomposition**). Given an endomorphism $\alpha \in R = \mathrm{End}(M_k)$ and another endomorphism $e \in \mathrm{idem}(R) \cap C(\alpha)$, we define $x := \alpha - e$ and say that the diagram

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & \downarrow{\scriptstyle \alpha\restriction_A = x\restriction_A} & & \downarrow{\scriptstyle (1-\alpha)\restriction_B = -x\restriction_B} \\ M & = & A & \oplus & B \end{array}$$

is the *ABAB-decomposition* corresponding to e.

When studying additive decompositions $\alpha = e + x$ (with $e^2 = e$ and $ex = xe$), one is naturally led to posit additional conditions on the element x. We recall a few common examples found in the literature. Throughout these examples we set $R := \mathrm{End}(M_k)$.

Example 1.3. **(Strongly clean elements).** Following [**23**], if $\alpha = e + x$ for some $e \in \mathrm{idem}(R) \cap C(\alpha)$, and additionally $x \in \mathrm{U}(R)$, we say this is a *strongly clean decomposition* for α, or that α is *strongly clean*. These elements arise in the study of exchange rings. Strongly clean decompositions for α are in one-to-one correspondence with ABAB-decomposition of the form

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & \downarrow{\scriptstyle \alpha\restriction_A = \mathrm{unit}} & & \downarrow{\scriptstyle (1-\alpha)\restriction_B = \mathrm{unit}} \\ M & = & A & \oplus & B \end{array}$$

noting that $(1-\alpha)\restriction_B = -x \restriction_B$ is a unit (i.e. an automorphism of B) if and only if $-(1-\alpha)\restriction_B = x \restriction_B$ is a unit.

Example 1.4. **(Strongly nil-clean elements).** Following [**9**], an element $\alpha \in R$ is *strongly nil-clean* if $\alpha = f + t$ for some $f \in \mathrm{idem}(R) \cap C(\alpha)$ and some $t \in \mathrm{nil}(R)$. Setting $e := 1 - f$ and $x := (1 - 2e + t)$, we see that $\alpha = e + x$, which for the purposes of this paper will be called a *strongly nil-clean complementary decomposition* for α. It is easy to compute that $-xe = (1-t)e$ and $x(1-e) = (1+t)(1-e)$. Recall that a ring element is said to be "unipotent" when it is of the form "1+nilpotent". Hence, the corresponding ABAB-decomposition looks like

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & \downarrow{\scriptstyle \alpha\restriction_A = \mathrm{unipotent}} & & \downarrow{\scriptstyle (1-\alpha)\restriction_B = \mathrm{unipotent}} \\ M & = & A & \oplus & B. \end{array}$$

Conversely, an ABAB-decomposition of this form corresponds to a strongly nil-clean complementary decomposition. If such a decomposition exists for α, then it is necessarily unique by [**14**, Theorem 3] or [**9**, Corollary 3.8].

EXAMPLE 1.5. **(Drazin invertible elements).** Again suppose $\alpha = e + x$ where $e \in \text{idem}(R) \cap C(\alpha)$ and $x \in \text{U}(R)$, as in the case of strongly clean elements. If additionally $\alpha e \in \text{nil}(R)$, then we say that α is *Drazin invertible.* (Such elements are also called *strongly π-regular* or *polar* in the literature.[1]) Equivalently, $-xe$ is unipotent in the corner ring eRe. In this case, such an equation $\alpha = e + x$ is *unique.* We'll call it the *Fitting decomposition* of α (in the ring R), and e is called the *spectral idempotent* of α. From these remarks, we see that $\alpha \in R = \text{End}\,(M_k)$ is Drazin invertible if and only if there is an ABAB-decomposition of the form

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & {\scriptstyle \alpha\restriction_A = \text{unit}}\big\downarrow & & \big\downarrow{\scriptstyle (1-\alpha)\restriction_B = \text{unipotent}} \\ M & = & A & \oplus & B. \end{array}$$

In fact, this is just the usual description of α being a Fitting endomorphism on the module M_k, and the equation $M = A \oplus B$ above is *also* called the Fitting decomposition of M (with respect to α).

We can generalize the previous example by replacing nilpotence with a more general property. Following Harte [**13**], an element $q \in R$ is called *quasi-nilpotent* if $1 - qr \in \text{U}(R)$ for every $r \in C(q)$, and we let $\text{qnil}(R)$ denote the set of all quasi-nilpotent elements of R. It is apparent that $\text{nil}(R) \cup J(R) \subseteq \text{qnil}(R)$. (The quasi-nilpotents are exactly the *topologically nil* elements of Banach algebras.) Just as an element $x \in R$ is unipotent when $x = 1 + t$ for some $t \in \text{nil}(R)$, we say that x is *quasi-unipotent* if $x = 1 + q$ for some $q \in \text{qnil}(R)$.

EXAMPLE 1.6. **(Quasipolar elements).** Generalizing Example 1.5, if $\alpha = e + x$ for some $e \in \text{idem}(R) \cap C(\alpha)$ with $x \in \text{U}(R)$ and $\alpha e \in \text{qnil}(R)$, then we call this a *quasi-Fitting decomposition* (or "generalized Fitting decomposition" modelling the terminology of [**6**]). Such a decomposition corresponds to an ABAB-decomposition of the form

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & {\scriptstyle \alpha\restriction_A = \text{unit}}\big\downarrow & & \big\downarrow{\scriptstyle (1-\alpha)\restriction_B = \text{quasi-unipotent}} \\ M & = & A & \oplus & B. \end{array}$$

If we strengthen the assumption that $e \in C(\alpha)$ to $e \in C^2(\alpha)$, then the element α is called *quasipolar.*[2] For any given $\alpha \in R$, there is at most one quasipolar representation, and hence at most one ABAB-decomposition as above with $e \in C^2(\alpha)$.

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[1]When defining polar elements, some authors instead of writing $\alpha = e + x$, use $\alpha + e = x$ (still with $e \in \text{idem}(R) \cap C(\alpha)$, $x \in \text{U}(R)$, and $\alpha e \in \text{nil}(R)$). Essentially this has the effect of replacing α with $-\alpha$ and x with $-x$.

[2]Here we are following the terminology in [**19**]. Many authors have previously used $\alpha + e = x$, rather than $\alpha = e + x$, to define quasipolarity. The reader should keep in mind that our renormalization will have payouts later in the element-wise bijections we construct.

EXAMPLE 1.7. **(Pseudopolar elements).** In the previous example, if we replace the condition "$\alpha e \in \mathrm{qnil}(R)$" with the weaker condition "there is some integer $n \geq 1$ such that $(\alpha e)^n \in J(R)$" then we call the corresponding decompositions *pseudo-Fitting*.

Further, replacing $e \in C(\alpha)$ with $e \in C^2(\alpha)$, the element α is called *pseudopolar*, and there is at most one such decomposition [**26**]. (Again, the comments in footnotes 1 and 2 apply.)

The ABAB-decompositions above make it clear we have the following containments:

$$\{\text{strongly nil-clean complementary decompositions}\} \subseteq \{\text{Fitting decompositions}\}$$
$$\subseteq \{\text{quasi-Fitting decompositions}\} \subseteq \{\text{pseudo-Fitting decompositions}\}$$
$$\subseteq \{\text{strongly clean decompositions}\}.$$

We will give one more (large) class of examples, and their connection to ABAB-decompositions.

DEFINITION 1.8 ($\mathcal{P}$**-Clean**). Let $\mathcal{P}$ be a property of elements in rings. Following [**9**], we say that $\alpha \in R$ is *strongly $\mathcal{P}$-clean* if $\alpha = e+x$ where $e \in \mathrm{idem}(R) \cap C(x)$, and $x \in R$ has $\mathcal{P}$. In this case we say that $\alpha = e + x$ is a *strongly $\mathcal{P}$-clean decomposition*, and that e is a *strongly $\mathcal{P}$-cleansing* idempotent for α.

DEFINITION 1.9 (**Strong Corner Property**). Let $\mathcal{P}$ be a property of elements in rings. We say that $\mathcal{P}$ is a *strong corner property* if for any $x \in R$ and any idempotent $e \in C(x)$, x has $\mathcal{P}$ in R if and only if exe and $(1-e)x(1-e)$ have $\mathcal{P}$ in eRe and $(1-e)R(1-e)$, respectively.

For instance, the property of being a unit, nilpotent, unipotent, or idempotent are all strong corner properties. We will see later (in Proposition 3.3) that so is being a quasi-nilpotent element.

PROPOSITION 1.10. *Let $\mathcal{P}$ be a strong corner property, and let $\alpha \in R$. The set of strongly $\mathcal{P}$-clean decompositions $\alpha = e + x$ is in natural bijection with the set of ABAB-decompositions of the form*

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & {\scriptstyle \alpha\restriction_A \text{ has } \mathcal{P}}\downarrow & & \downarrow{\scriptstyle -(1-\alpha)\restriction_B \text{ has } \mathcal{P}} \\ M & = & A & \oplus & B. \end{array}$$

PROOF. This follows by the proof of [**9**, Lemma 2.8], *mutatis mutandis.* □

2. Bott-Duffin Decompositions

In the theory of generalized inverses in rings, there is one very useful kind of invertibility that is known as Bott-Duffin invertibility. For a given idempotent e in a ring R, an element $x \in R$ is said to be *Bott-Duffin invertible relative to e* if $exe \in \mathrm{U}(eRe)$. In this case, the inverse of exe in the corner ring eRe is known as the *Bott-Duffin inverse* of x (relative to e). Making use of these definitions, we now introduce the main object of study in this section, which is a special class of additive decompositions of elements in a ring based on the notion of Bott-Duffin invertibility.

Definition 2.1 (**Bott-Duffin Decomposition**). Let R be a ring and $\alpha \in R$. If $\alpha = e + x$ where $e \in \text{idem}(R) \cap C(x)$ and x is Bott-Duffin invertible relative to e, we call this a *Bott-Duffin decomposition* of α. We will say that e is the *spectral idempotent* of the Bott-Duffin decomposition $\alpha = e + x$.

Note that since $ex = xe$ in Definition 2.1, the condition for x to be Bott-Duffin invertible relative to e simply boils down to $xe \in \text{U}(eRe)$. Thus, we'll be using the element-wise notion of Bott-Duffin invertibility only in a rather easy special case. In this case, the Bott-Duffin inverse of x relative to e (if it exists) is simply the unique element $v \in eRe$ such that $v(ex) = (ex)v = e$, or equivalently, just $vx = xv = e$. The example below illustrates how in the simplest cases Bott-Duffin decompositions can behave in rings.

Example 2.2. Every element $\alpha \in R$ has at least one Bott-Duffin decomposition, namely $\alpha = 0 + \alpha$, but in general there may exist many other decompositions. The element $0 \in R$ has a Bott-Duffin decomposition $0 = e + (-e)$ for every idempotent $e \in R$. On the other hand, the element $1 \in R$ has only the trivial Bott-Duffin decomposition $1 = 0 + 1$. Any strongly clean decomposition $\alpha = e + x$ is clearly a Bott-Duffin decomposition (since $x \in \text{U}(R) \cap C(e)$ has Bott-Duffin inverse $x^{-1}e$ relative to e). In particular, any idempotent $\varepsilon \in R$ has a Bott-Duffin decomposition $\varepsilon = (1 - \varepsilon) + (2\varepsilon - 1)$ (with spectral idempotent $1 - \varepsilon$).

We have also a close relationship between the Bott-Duffin decompositions of an endomorphism and certain ABAB-decompositions, as in the following result which is the special case of Proposition 1.1 with the added requirement that $xe \in \text{U}(eRe)$.

Proposition 2.3. *Given $\alpha \in R = \text{End}(M_k)$, the Bott-Duffin decompositions $\alpha = e + x$ are in one-to-one correspondence with diagrams*

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & {\scriptstyle \alpha\restriction_A}\big\downarrow & & \big\downarrow{\scriptstyle (1-\alpha)\restriction_B = \text{unit}} \\ M & = & A & \oplus & B. \end{array}$$

Note that in this diagram, the left vertical arrow labelled $\alpha\restriction_A$ is intended to mean just that the summand A is α-invariant, with no other conditions attached. This is due to the fact that, if A is taken to be $\ker(e)$, then $\alpha\restriction_A = x\restriction_A$, which is *not* subject to any requirements in Definition 2.1. We note also that the right vertical arrow labelled $(1-\alpha)\restriction_B$ is only required to be an automorphism of B, with again no other conditions attached. Therefore, the decomposition examples given in (1.3)–(1.7) in §1 enable us to draw the following useful conclusions. (In retrospect, of course, it was these conclusions that have provided us the main motivation for introducing the notion of Bott-Duffin decompositions in Definition 2.1.)

Corollary 2.4. *Strongly clean, Fitting, quasi-Fitting, pseudo-Fitting, strongly unipotent-clean, and strongly nil-clean complementary decompositions are Bott-Duffin decompositions.*

We next prove the first main theorem of this paper, which connects the Bott-Duffin decompositions for the two elements in a Jacobson pair (α, β). The main point of this theorem is that it introduces a procedure whereby a Bott-Duffin decomposition of α will give rise to a "corresponding" Bott-Duffin decomposition of

β. Furthermore, when the same procedure is repeated on the latter Bott-Duffin decomposition, one gets back the original Bott-Duffin decomposition of α.

THEOREM 2.5. *Let $a, b \in R$, $\alpha := 1 - ab$, and $\beta := 1 - ba$. Suppose that $\alpha = e + x$ is a Bott-Duffin decomposition in R, and let v denote the Bott-Duffin inverse of x relative to e.*

(A) *The element $f := -bva$ is an idempotent that is isomorphic to e. (Equivalently, $1-e$ forms a Jacobson pair with $1-f$.) Also, we have $ReR = RfR$.*
(B) *If $y := \beta - f$, then (x, y) is a Jacobson pair. The commutation relations $\alpha a = a\beta$ and $b\alpha = \beta b$ are inherited by e and f; that is, $ea = af$, $be = fb$, and also, $xa = ay$, $bx = yb$.*
(C) *The equation $\beta = f + y$ is a Bott-Duffin decomposition, where y has Bott-Duffin inverse $-bv^2a$ relative to f; that is, $(fy)^{-1} = -bv^2a$ in the corner ring fRf.*
(D) *We have $e = -a\,(fy)^{-1}b$. Thus, if we apply the same procedure starting with the Bott-Duffin decomposition $\beta = f + y$ (and switching the roles of a and b), we will get back $\alpha = e + x$. In particular, our procedure gives a bijective correspondence between the Bott-Duffin decompositions of α and those of β.*
(E) *$(1 - e + xe,\ 1 - f + yf)$ is a Jacobson pair in R, and so is $(1 - e - xe,\ 1 - f - yf)$.*
(F) *For any integer $n \geq 0$ and any ideal $J \subseteq R$, $(\alpha e)^n \in J$ iff $(\beta f)^n \in J$.*
(G) *For any ideal $J \subseteq R$, $\overline{\alpha e}$ is quasi-nilpotent in R/J iff $\overline{\beta f}$ is quasi-nilpotent in R/J.*
(H) (Double-Centralizer Property) *$e \in C^2(\alpha)$ iff $f \in C^2(\beta)$.*

PROOF. (A) Left multiplying $ab = 1 - e - x$ by $v = ve$ gives $vab = -vx = -e$, and similarly, $abv = -e$. Thus,

$$f^2 = (-bva)(-bva) = b(vab)va = -beva = -bva = f.$$

By [**18**, Theorem 21.20], the idempotent $f = (-bv)\,a$ is isomorphic to the idempotent $e = a\,(-bv)$. The equation $f = -beva$ shows that $RfR \subseteq ReR$. After proving (D) below, we can likewise conclude that $ReR \subseteq RfR$, so we have in fact $ReR = RfR$.

(B) We have $ea = -abva = af$, and similarly, $be = fb$. Combining these with $\alpha a = a\beta$ and $b\alpha = \beta b$, we get $xa = ay$ and $bx = yb$. Clearly, $x = \alpha - e = 1 - a\,(b - bv)$ forms a Jacobson pair with $1 - (b - bv)\,a = 1 - ba + bva = \beta - f = y$.

(C) To begin with, $f\beta = f - fba = f - bea = f - baf = \beta f$. This implies that $fy = yf$. The rest of the proof of part (C) will follow if we can show that $-bv^2a \in fRf$ and that it provides an inverse for fy in fRf. The fact that $-bv^2a \in fRf$ follows from $f\,(bv^2a) = bev^2a = bv^2a$ and a similar equation $(bv^2a)f = bv^2a$. Left multiplying the former by $-y$ (which commutes with f) gives

$$(fy)\,(-bv^2a) = -ybv^2a = -bxv^2a = -bva = f,$$

and a similar computation gives $(-bv^2a)\,(fy) = f$.

(D) Using (C) and the equations $e = -abv = -vab$, we have

$$-a\,(fy)^{-1}b = -a\,(-bv^2a)\,b = (abv)\,(vab) = (-e)^2 = e.$$

Noting that $(fy)^{-1}$ is the Bott-Duffin inverse of y relative f, the rest of (D) clearly follows.

(E) Since $1 - e + xe = 1 + vab - xvab = 1 + (va - xva)\, b$, it forms a Jacobson pair with

$$1 + b\,(va - xva) = 1 - f - bxva = 1 - f - ybva = 1 - f + yf.$$

Clearly, the same calculation works for the pair $(1 - e - xe, 1 - f - yf)$.

(F) It suffices to prove the "only if" statement, so assume that $(\alpha e)^n \in J$. By repeated use of the commutation rule $\beta b = b\alpha$, we have

$$(\beta f)^n = \beta^n f = \beta^n(-bva) = -b\alpha^n(ev)\, a \in R\,(\alpha e)^n R \subseteq J.$$

(G) After replacing R by R/J, we may assume that $J = 0$. Proceeding as in (F), we note that $\beta f = \beta\,(-bva) = -b\,(\alpha va)$, while $-(\alpha va)\, b = \alpha\,(-vab) = \alpha e$. Thus, we see that (G) follows from the fact that, for any two elements $s, t \in R$, st is quasi-nilpotent iff ts is quasi-nilpotent; see [**7**]. (We will actually re-prove this fact as a part of Proposition 3.3 below.)

(H) This is possibly the most tricky part in this theorem. Assuming that $e \in C^2(\alpha)$, consider any element $r \in C(\beta)$. *Our goal is to show that* $r \in C(f)$. To begin with, note that $r \in C(ba)$, so we have an equation $rba = bar$. Using this, we get $ab\,(arb) = a\,(rba)\,b = arb\,(ab)$. Therefore, $arb \in C(ab) = C(\alpha)$, which implies that $arb \in C(e)$ and hence also $arb \in C(x)$. This implies that arb also commutes with the Bott-Duffin inverse of x relative to e, that is, $arb \in C(v)$, either by a direct computation or appealing to Drazin's commutation theorem in [**11**, Theorem 2.3]. Now consider the element bv^2arba, which we'll compute in two different ways. First,

$$bv^2a\,(rba) = bv^2a\,(bar) = bv\,(-e)\,ar = -bvar = fr.$$

Next, commuting arb twice with v, we get

$$bv^2(arb)\,a = b\,(arb)\,v^2a = (rba)\,bv^2a = -rbeva = -rbva = rf.$$

These computations show that $r \in C(f)$, as desired. The converse follows by symmetry. □

REMARK 2.6. The significance of the fact proved in (E) is as follows. Since $xe \in \mathrm{U}(eRe)$, we have $1 - e + xe \in \mathrm{U}(R)$. Therefore, (E) implies that $1 - f + yf \in \mathrm{U}(R)$. From this (and the fact that $yf = fy$), it is *also* easy to show that $fy \in \mathrm{U}(fRf)$. Moreover, using the Desert Island Formula for the Jacobson pairs in (E), it is possible to compute *explicitly* $(fy)^{-1}$. In fact, in our work, this was how we first arrived at the formula $(fy)^{-1} = -bv^2a$ in (C). But just as in the case of the Desert Island Formula, once this "answer" is known, it is very easy to verify by an explicit calculation that the answer does work. Since this "method of proof" is a bit easier to write down, we have for convenience chosen to follow it in the proof of (C) above, instead of first digressing to prove the fact (E).

REMARK 2.7. A few words are perhaps in order to explain why we required that $ex = xe$ in Definition 2.1. If we define $\alpha = e + x$ to be a "general Bott-Duffin decomposition" by assuming only $e^2 = e$ and that x is Bott-Duffin invertible relative to e, then we can no longer expect any one-to-one correspondence between the "general Bott-Duffin decompositions" of α and those of β. For an explicit example to illustrate this, let $R = \mathbb{M}_2(S)$ where S is any nonzero ring. Let $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

and $b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in R. Then $ab = 0$, and $\alpha := I_2 - ab = I_2$ is easily seen to have only one "general Bott-Duffin decomposition" (namely, $0 + I_2$). However, $\beta := I_2 - ba = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in \mathrm{U}(R)$ has (at least) the following family of "general Bott-Duffin decompositions":

$$\beta = 0 + \beta = \begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -s \\ -1 & 0 \end{pmatrix} \text{ (for any } s \in \mathrm{U}(S)).$$

Here, the last matrix is Bott-Duffin invertible relative to $\begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix}$, with Bott-Duffin inverse $\begin{pmatrix} 0 & -1 \\ 0 & -s^{-1} \end{pmatrix}$. (In particular, β could have as many "general Bott-Duffin decompositions" as we want, by suitably choosing S.) Nevertheless, according to Theorem 2.5, the trivial decomposition $\beta = 0 + \beta$ is the *only* Bott-Duffin decomposition of β (in the sense of Definition 2.1). We note, incidentally, that all decompositions of β listed above are clean decompositions.

We close this section with a result on the case where $ab = ba$. In this case, $\alpha = \beta$, so it would be of interest to see which Bott-Duffin decompositions $\alpha = e + x$ correspond to themselves under the one-to-one correspondence given in Theorem 2.5. The answer to this question is given as follows.

Proposition 2.8. *Let $a, b \in R$ be such that $ab = ba$, and let $\alpha := 1 - ab = e + x$ be a Bott-Duffin decomposition, with a corresponding decomposition $\beta := 1 - ba = f + y$ as in* Theorem 2.5*. Then $e = f$ iff $e \in C(a)$, iff $e \in C(b)$.*

Proof. First assume that $e = f$. By Theorem 2.5(B), we have $ea = af = ae$ and $be = fb = eb$, so $e \in C(a) \cap C(b)$. Conversely, assume that $e \in C(a)$. Since $b \in C(a)$, we have $x = 1 - ab - e \in C(a)$ too. As in the proof of Theorem 2.5, let $v \in eRe$ be the Bott-Duffin inverse of x relative to e. Then Drazin's commutation theorem [**11**, (2.3)] shows (as before) that $v \in C(a)$. Recalling that $e = -abv$ (from the proof of Theorem 2.5(A)), we have $f := -bva = -bav = -abv = e$. If we assume, instead, that $e \in C(b)$, a similar calculation shows that $v \in C(b)$, so again $f = -bva = -vba = -vab = e$. (We note that, a priori, it was not at all clear that $e \in C(a)$ would imply that $e \in C(b)$, or vice versa.) □

Example 2.9. Instead of working with a Jacobson pair $(1 - ab, 1 - ba)$, one may also work directly with the pair (ab, ba) too. In this case, however, the Bott-Duffin decompositions of ab may no longer be in one-to-one correspondence with those of ba. For instance, if $a, b \in R$ are such that $ab = 1 \neq ba$, then ab has only one Bott-Duffin decomposition $ab = 0 + 1$. On the other hand, ba has at least two Bott-Duffin decompositions; namely, $ba = 0 + ba = (1 - ba) + (2ba - 1)$.

3. Preserving More Properties

The bijection we constructed in §2 does more than simply preserve the underlying Bott-Duffin structure. Indeed, as we will see shortly, the following very general definition will lead to a large class of ring theoretic decomposition properties that are also preserved by the mapping constructed in Theorem 2.5.

Definition 3.1 (**Strong Corner Jacobson Property**). Let $\mathcal{P}$ be a condition on elements of rings. We say that $\mathcal{P}$ is a *strong corner Jacobson property* if the following three conditions hold.

(i) $\mathcal{P}$ is a "Jacobson property", in the sense that, if (α, β) is a Jacobson pair in R and α has $\mathcal{P}$ in R, then β has $\mathcal{P}$ in R.
(ii) The property $\mathcal{P}$ is a strong corner property, in the sense of Definition 1.9.
(iii) The element $1 \in R$ always has $\mathcal{P}$ in R.

Example 3.2. Being a unit is a strong corner Jacobson property by an application of Jacobson's Lemma. Somewhat easier is the fact that *unipotence* is also a strong corner Jacobson property. Indeed, if $1 - xy$ is unipotent, then xy is nilpotent, and so is yx (with the index of nilpotence changing by at most one). This means that $1 - yx$ is indeed unipotent. (The remaining properties among (i), (ii) and (iii) above are all easy to check.) At this point, it would be natural to ask whether the same fact holds if unipotence is replaced by *quasi-unipotence.* We'll supply a positive answer to this question below as we have not been able to find a full reference for the following result in the literature.

Proposition 3.3. *Quasi-unipotence is a strong corner Jacobson property.*

Proof. It turns out that a fairly substantial argument is needed. Again, to check that quasi-unipotence is a Jacobson property rests on checking that

(∗) xy being quasi-nilpotent implies that yx is quasi-nilpotent.

This has been done, for instance, in [**7**] (for Banach algebras). For completeness, we'll include a proof here for general rings. To show that yx remains quasi-nilpotent, consider any element $r \in R$ with $r\,(yx) = (yx)\,r$. Then xr^2y commutes with xy since

$$(xr^2y)xy = xr^2(yx)y = x(yx)r^2y = xy(xr^2y).$$

From the quasi-nilpotence of xy, we have $1 - (xr^2y)xy \in \mathrm{U}(R)$. By Jacobson's Lemma,

$$1 - y(xr^2y)x = 1 - (yxr)(ryx) = 1 - (yxr)\,(yxr) = (1 - yxr)(1 + yxr) \in \mathrm{U}(R).$$

Since $1 - yxr$ and $1 + yxr$ commute, this proves that $1 + yxr \in \mathrm{U}(R)$, and hence yx is quasi-nilpotent.

Since (3.1)(iii) trivially holds, it remains only to show that, if $e = e^2 \in R$ and $z \in R$ are such that $ez = ze$, then z is quasi-unipotent in R iff eze is quasi-unipotent in eRe and $e'ze'$ is quasi-unipotent in $e'Re'$, where $e' := 1 - e$. The "only if" part is easy. For the "if" part, assume the stated properties on eze and $e'ze'$, say $eze = e + q$ and $e'ze' = e' + q'$ for some quasi-nilpotent elements $q \in eRe$ and $q' \in e'Re'$. We need to show that $q + q'$ is quasi-nilpotent in R. For the remainder of this proof, we will freely write elements of R in the form of 2×2 matrices via their Peirce decompositions with respect to the idempotent e. Thus, for instance, we write $q + q' = \begin{pmatrix} q & 0 \\ 0 & q' \end{pmatrix}$. Consider any $r = \begin{pmatrix} s & t \\ t' & s' \end{pmatrix} \in R$ that commutes with $q + q'$; that is, $sq = qs$, $s'q' = q's'$, $t'q = q't'$, and $tq' = qt$. We must show that $1 + r(q + q') \in \mathrm{U}(R)$. Write

$$1 + r\,(q + q') = \begin{pmatrix} e + sq & tq' \\ t'q & e' + s'q' \end{pmatrix}. \tag{3.4}$$

Since $sq = qs$ and $q \in eRe$ is quasi-nilpotent, $e + sq$ is a unit in eRe, say with inverse u; and similarly $e' + s'q'$ has inverse v in $e'Re'$. As $e + sq$ commutes with q, so does u; that is, $uq = qu$. Similarly, we have $vq' = q'v$. Multiplying (3.4) on the left by the unit $\begin{pmatrix} e & 0 \\ -vt'q & e' \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, it suffices to show that

$$\begin{pmatrix} e & utq' \\ 0 & e' - vt'qutq' \end{pmatrix} \in \mathrm{U}(R). \tag{3.5}$$

Using the four commuting relations $t'q = q't'$, $tq' = qt$, $vq' = q'v$ and $uq = qu$, we have

$$(vt'qut)\,q' = vq't'uqt = q'(vt'qut).$$

As q' is quasi-nilpotent in fRf, this implies that the lower-right corner of the matrix in (3.5) is a unit in fRf, and so the whole matrix is a unit in R, as desired. □

The presentation of the main result in this section depends crucially on the next proposition on the transfer of a strong corner Jacobson property "from $\pm ex$ to $\pm fy$" for a corresponding pair of Bott-Duffin decompositions in the notations of Theorem 2.5.

PROPOSITION 3.6. *Let $\mathcal{P}$ be a strong corner Jacobson property. Let $(\alpha, \beta) = (1 - ab, 1 - ba)$ be a Jacobson pair, and let $\alpha = e + x$ and $\beta = f + y$ be a pair of corresponding Bott-Duffin decompositions under the bijection given in Theorem* 2.5.

(A) *If $ex = e\,(\alpha - 1)$ has $\mathcal{P}$ in eRe, then $fy = f(\beta - 1)$ has $\mathcal{P}$ in fRf.*
(B) *If $-ex = e\,(1 - \alpha)$ has $\mathcal{P}$ in eRe, then $-fy = f(1 - \beta)$ has $\mathcal{P}$ in fRf.*
(C) *If $(1 - e)\,x = (1 - e)\,\alpha$ has $\mathcal{P}$ in $(1 - e)R\,(1 - e)$, then $(1 - f)\,y = (1 - f)\beta$ has $\mathcal{P}$ in $(1 - f)R\,(1 - f)$.*

PROOF. To simplify the notations, let $e' = 1 - e$ and $f' = 1 - f$.

(A) Suppose that ex has $\mathcal{P}$ in eRe. The element $e' = 1_{e'Re'}$ has $\mathcal{P}$ in $e'Re'$ by (3.1)(iii), and hence $e' + ex$ has $\mathcal{P}$ in R by (3.1)(ii). By Theorem 2.5(E), $e' + ex$ and $f' + fy$ form a Jacobson pair. Hence $f' + fy$ has $\mathcal{P}$ in R by (3.1)(i), and hence so does fy in fRf by (3.1)(ii).

(B) This case is not directly covered by case (A) since elements satisfying property $\mathcal{P}$ are not necessarily closed under negation. However, the argument used for case (A) still works here, as Theorem 2.5(E) showed that $(e' - ex,\ f' - fy)$ also form a Jacobson pair.

(C) The argument here is again similar to that used for proving (A), but with e and e' playing reversed roles. Assume that $e'x$ has $\mathcal{P}$ in $e'Re'$. As $e = 1_{eRe}$ has $\mathcal{P}$ in eRe by (3.1)(iii), so does $e + e'x$ in R by (3.1)(ii). Noting that $fy = f(f' - ba) = -fba = -bea$ (by Theorem 2.5(B)), we see that

$$e + e'x = e + e'\alpha = \alpha + e - e\alpha = 1 - ab + eab$$

forms a Jacobson pair with

$$1 - ba + bea = \beta - fy = f + y - fy = f + f'y.$$

Thus, $f + f'y$ has $\mathcal{P}$ in R by (3.1)(i), and hence so does $f'y$ in $f'Rf'$ by (3.1)(ii). □

We are now ready to prove the second main result of this paper.

THEOREM 3.7. *The one-to-one correspondence in Theorem* 2.5 *respects strongly clean, Fitting, quasi-Fitting, pseudo-Fitting, strongly unipotent-clean, and strongly nil-clean complementary decompositions.*

Proof. First of all, the various kinds of decompositions listed above are all Bott-Duffin decompositions by Corollary 2.4. Furthermore, all of the needed conditions on ex, $-ex$, or $(1-e)\,x$ are strong corner Jacobson properties, as we can see by applying (1.10), (3.2), and (3.3) to Examples (1.3)–(1.7). All desired conclusions now follow from this observation and Proposition 3.6. □

Certainly, the result above covers some recently known cases of properties that are preserved by the passage from $\alpha = 1 - ab$ to $\beta = 1 - ba$. This includes, for instance, the case where $1 - ab$ is Drazin invertible (as proved in [**7, 24, 21**]), and more generally, the case where $1 - ab$ is quasi-Drazin invertible (as proved in [**27**]). Strongly nil clean elements were handled recently in [**17**, Theorem 2.9 and Corollary 2.10]. According to Theorem 3.7 above, we now also know that the same result holds for a number of other cases; for instance, if $1 - ab$ is strongly clean, or uniquely strongly clean, then so is $1 - ba$.[3] In a way, it is no longer surprising that such results *do not* apply, for instance, to clean elements since clean decompositions for $\alpha = 1 - ab$ need not be Bott-Duffin decompositions (in the sense of Definition 2.1). Taking the "uniquely strongly clean" case, we conclude this section with the following nontrivial application of Theorem 3.7.

Corollary 3.8. *If $ab \in R$ is an idempotent, then $\beta := 1 - ba \in R$ is uniquely strongly clean. It is also strongly regular, with Fitting decomposition $\beta = (ba)^2 + (1 - ba - (ba)^2)$.*

Proof. Since $\alpha := 1 - ab$ is an idempotent, it has a unique strongly clean decomposition (namely, $\alpha = e + x$ with $e = 1 - \alpha = ab$ and $x = 1 - 2ab$). Applying Theorem 3.7, we see that the same holds for $\beta = 1 - ba$. To work out the strongly clean decomposition for β, we note that x has Bott-Duffin inverse ex^{-1} relative to e. By the proof of Theorem 2.5, the "corresponding" decomposition $\beta = f + y$ is supposed to have

$$f = -b\,((ab)\,x^{-1})\,a = -bab\,(1 - 2ab)\,a = -(ba)^2 + 2b\,(ab)^2 a = (ba)^2.$$

Therefore, $y = \beta - f = 1 - ba - (ba)^2$. (Incidentally, y^{-1} is given by $1 + ba - 3\,(ba)^2$.) Finally, the fact that α is strongly regular implies that β is also strongly regular by [**21**, Corollary 2.7], so $\beta = f + y$ is necessarily the Fitting decomposition of β. □

4. Mock Drazin Inverses for Strongly Clean Decompositions

In this section, we shall look more closely at what happens to the results in §2 and §3 in the special case where the property $\mathcal{P}$ is taken to be "being a unit" (in a ring). In this case, a decomposition $\alpha = e + x$ subject to the requirements $e = e^2$ and $x \in \mathrm{U}(R) \cap C(e)$ would simply be a *strongly clean decomposition* of an element $\alpha \in R$. As we have stated in Example 2.2, such a decomposition is always a Bott-Duffin decomposition. In general, if $\alpha = e + x$ is a Bott-Duffin decomposition, the "complementary equation" $1 - \alpha = (1 - e) + (-x)$ need not be Bott-Duffin decomposition. As an easy exercise, the reader can check that $1 - \alpha = (1 - e) + (-x)$ is *also* a Bott-Duffin decomposition iff $x \in \mathrm{U}(R)$; that is, iff $\alpha = e + x$ is a strongly clean decomposition. The main goal of this section

[3]The fact that $1 - ab$ being strongly clean implies the same for $1 - ba$ was announced by the authors in [**21**] in 2012, but had so far remained unpublished. A (different) proof of this result by Gürgün has now appeared in [**12**].

will be to give more detailed information in this important special case. The extra leverage in this case is that we can introduce the interesting notion of the *mock Drazin inverse* of such a decomposition, as follows.

DEFINITION 4.1 (**Mock Drazin Inverse**). Let $\alpha = e+x$ be any strongly clean decomposition. As $x \in \mathrm{U}(R)$, we define the element $\alpha' := (1-e)\,x^{-1}$ to be the **mock Drazin inverse** of the given decomposition $\alpha = e+x$. (Note that, although we have used here the convenient notation α' for the mock Drazin inverse, it *does* depend on the choice of the decomposition $\alpha = e+x$, and is not determined by the element α alone.)

To motivate the above definition, we recall that, in the classical case where $\alpha \in R$ is *polar* (i.e. Drazin invertible) with (unique) Fitting decomposition $\alpha = e+x$ as defined in Example 1.5, the standard Drazin inverse of α is given by the element $(1-e)\,x^{-1}$; see e.g. [**16**, Theorem 4.2]. In general, an element $\alpha \in R$ *may not* be polar, but it may still have a strongly clean decomposition $\alpha = e+x$. In this case, it would therefore be useful to study the mock Drazin inverse $(1-e)\,x^{-1}$ associated with such a decomposition of α. Fortuitously, it turns out that some of the facts about Drazin invertible elements in rings can be extended to such a more general framework. To begin with, note that

$$\alpha\alpha' = (e+x)\,(1-e)\,x^{-1} = x\,(1-e)\,x^{-1} = 1-e, \tag{4.2}$$

which may be called the *associated idempotent* of the decomposition $\alpha = e+x$. From (4.2), we see easily that $\alpha'\alpha\alpha' = \alpha'$, which is the first axiomatic property of the classical Drazin inverse. Next, for any integer $n \geq 0$, we have

$$\alpha^n - \alpha^{n+1}\alpha' = \alpha^n(1-\alpha\alpha') = \alpha^n e. \tag{4.3}$$

In order that α is polar and that $e+x$ is its Fitting decomposition, the extra condition needed is that $(\alpha e)^n = \alpha^n e = 0$ for some integer $n \geq 1$ (see, e.g. [**16**]), which, by (4.3) above, amounts precisely to the second axiomatic property $\alpha^n = \alpha^{n+1}\alpha'$ for α' to be the classical Drazin inverse of α. The smallest integer $n \geq 1$ for which $(\alpha e)^n = 0$ (that is, the nilpotence index of αe) is known as the *Drazin index* of α. In the case of *quasipolar* or *pseudopolar* elements, we have basically the same characterizations, except that in the quasipolar case we require that αe is quasi-nilpotent, and in the pseudopolar case we require that some power of αe lies in the Jacobson radical $J(R)$. The quasi-Drazin inverse and the pseudo-Drazin inverse of α are still given by the same expression $(1-e)\,x^{-1}$ arising from a (unique) quasi-Fitting or pseudo-Fitting decomposition $\alpha = e+x$; see [**16**], [**27**], and [**26**]. In our more general set-up, however, $\alpha = e+x$ can be *any* strongly clean decomposition of the element α. The point of Definition 4.1 is that a mock Drazin inverse can be defined for such a decomposition, without requiring any property to be satisfied by the element $\alpha e = e\alpha$.

For any Jacobson pair (α, β), we have carried out a study in our earlier paper [**21**] on how the Fitting decompositions (and Drazin inverses) of α and β are related, when α is assumed to be polar. Our next goal now is to prove the third main result of this paper, which is to extend a number of our earlier results in [**21**] to the case of *general* strongly clean decompositions of α and β, as follows.

THEOREM 4.4. *Let* $(\alpha, \beta) = (1-ab, 1-ba)$ *be a Jacobson pair, and let* $\alpha = e+x$ *and* $\beta = f+y$ *be a pair of "corresponding" strongly clean decompositions* (*in the sense of* Theorem 3.7). *Then the following conclusions hold.*

(A) *$f = -b\,(ex^{-1})\,a$ and $y^{-1} = 1 + b\,(x^{-1} - ex^{-2})\,a$.*

(B) *$\alpha' = 1 + abx^{-1}$, $\beta' = 1 + bay^{-1}$, and (α', β') is a Jacobson pair satisfying the commutation relations $\alpha' a = a\beta'$ and $b\alpha' = \beta' b$.*

(C) *The mock Drazin inverse of β associated with the strongly clean decomposition $\beta = f + y$ is given by $\beta' = 1 - f + b\,\alpha' a$.*

(D) *If $a, b \in \mathrm{U}(R)$ or R is an* IC *ring (in the sense of* [**15**]*), the two spectral idempotents e and f are similar.*

(E) *Suppose α is polar, with Fitting decomposition $\alpha = e + x$. Then β is also polar (with the same Drazin index), and the corresponding strongly clean decomposition $\beta = f + y$ is the Fitting decomposition of β. Finally, we have $f = bera$ where $r = (1 - \alpha e)^{-1}$.*

(F) *Without the Drazin index part,* (E) *above holds also with "polar" replaced by "quasipolar" (resp. "pseudopolar") and "Fitting" replaced by "quasi-Fitting" (resp. "pseudo-Fitting").*

Proof. We shall use freely the notations and formulas in the proof of Theorem 2.5. Here, the Bott-Duffin inverse of x relative to e is simply given by $v := ex^{-1} \in \mathrm{U}(eRe)$.

(A) Recalling Theorem 2.5(A), we have $f = -bva = -b\,(ex^{-1})\,a$. Next, $x = 1 - a\,(b - bv)$ forms a Jacobson pair with $y = 1 - (b - bv)\,a$. Since $x \in \mathrm{U}(R)$, it follows that $y \in \mathrm{U}(R)$, and the Desert Island Formula gives

$$(\dagger) \qquad y^{-1} = 1 + (b - bv)\,x^{-1}a = 1 + b\,(x^{-1} - ex^{-2})\,a.$$

This may be thought of as a higher form of the Desert Island Formula (with an extra "quadratic term"). Indeed, in the case where $e = 0$, we'll have $f = 0$ too since f is isomorphic to e. In this situation, $x = \alpha = 1 - ab$ and $y = \beta = 1 - ba$, in which case ($\dagger$) reverts to the classical Desert Island Formula.

(B) The equation giving the mock Drazin inverse α' follows from

$$\alpha' := (1 - e)\,x^{-1} = (x + ab)\,x^{-1} = 1 + abx^{-1},$$

and similarly, $\beta' = 1 + bay^{-1}$. Recalling the commutation rule $xa = ay$ from Theorem 2.5(B), we have $ay^{-1} = x^{-1}a$. Thus, $\beta' = 1 + bx^{-1}a$, which forms a Jacobson pair with $1 + abx^{-1} = \alpha'$. Finally, the other commutation rule $ea = af$ from Theorem 2.5(B) gives

$$\alpha' a = (1 - e)\,x^{-1}a = (1 - e)\,ay^{-1} = a\,(1 - f)\,y^{-1} = a\beta',$$

and a similar argument gives $b\alpha' = \beta' b$.

(C) Using (A), (B), and the fact that $ay^{-1} = x^{-1}a$, we have

$$\begin{aligned} \beta' &= 1 + bay^{-1} = 1 + bx^{-1}a = 1 + b\,[e + (1 - e)]\,x^{-1}a \\ &= 1 + bex^{-1}a + b\,(1 - e)\,x^{-1}a = 1 - f + b\,\alpha' a, \end{aligned}$$

(D) If R is an IC ring, the isomorphic idempotents e, f must be similar by [**15**, Corollary 7.6]. For the other case, recall from the proof of Proposition 3.6(C) that $fy = -bea$. If we assume that $a, b \in \mathrm{U}(R)$, then since $y \in \mathrm{U}(R)$ too, the idempotent f is *equivalent* to e. (Two idempotents e_1, e_2 are called equivalent if $e_1 = ue_2w$ for some $u, w \in \mathrm{U}(R)$.) By a theorem of Song and Guo [**25**, Theorem 2], e and f are similar in R. (Note that, in general, e and f need not be similar. For instance, suppose $ab = 1 \neq ba$. Then $\alpha = 1 - ab = 0$ has a unique strongly clean decomposition $1+(-1)$, with $e = 1$, while the idempotent $\beta = 1 - ba$ has a unique strongly clean decomposition $f + (1 - 2\,ba)$, with $f = ba$ not similar to $e = 1$.)

(E) All conclusions in (E) follow from Theorem 2.5 and Theorem 3.7, except for the fact that $f = -b\,(ex^{-1})\,a$ can now be written in the new form $f = bera$ where $r = (1-\alpha e)^{-1}$. To see this, let $t := \alpha e = e + ex$, which is nilpotent. Since $e\,(1-t) = e - \alpha e = -ex$, we have $ex^{-1} = -e\,(1-t)^{-1} = -er$, so $f = -b\,(ex^{-1})\,a = bera$. (The advantage of this new formula is that it expresses f directly in terms of a, b and the spectral idempotent e of α without explicitly using x or x^{-1}. If α has Drazin index n, then $t^n = 0$ and $r = 1 + t + \cdots + t^{n-1}$, so $f = bera$ can further be written as $f = besa$ where $s = 1 + \alpha + \cdots + \alpha^{n-1}$. We also note that $1 - f = \beta\beta'$, the associated idempotent of β, can be expressed in the form $(1 - bea)^m$ for any $m \geq n$. This fact was fully proved in [**21**, Theorem 2.1].)

(F) The same proof for (E) essentially carries over verbatim. Here, instead of being nilpotent, αe is quasi-nilpotent (resp. has a power lying in $J(R)$), so the inverse $r = (1 - \alpha e)^{-1}$ exists nevertheless. From part (F) and part (G) of Theorem 2.5, it follows that βf remains quasi-nilpotent (resp. has a power lying in $J(R)$). The only other thing to be careful about is that we have to ensure that the double-centralizer property $f \in C^2(\beta)$ holds so that $f + y$ is a quasi-Fitting (resp. pseudo-Fitting) decomposition of β. This follows from Theorem 2.5(H) since we have $e \in C^2(\alpha)$ to begin with. □

In the case where $\alpha = 1 - ab$ is polar, the equation in part (C) relating the mock Drazin inverse of β to that of α boils down to the Drazin inverse formula for Jacobson pairs obtained in [**21**, Theorem 2.1], and the equation $f = bera$ in part (E) recovers [**21**, Theorem 2.4]; see also [**3**, Theorem 3.6].[4] In the case where α is quasipolar (resp. pseudopolar), part (C) and part (F) recover the results of Zhuang, Chen and Cui in [**27**] and Cvetković-Ilić and Harte in [**7**] (resp. those of Gürgün in [**12**]). Our treatment here is, however, much more efficient, in that one single short proof is used to get the results in all three cases. Moreover, we have shown that the first four parts of Theorem 4.4 hold quite generally for any corresponding pair of strongly clean decompositions of α and β.

In the case where *all* elements of a ring R are polar, we can draw another more precise conclusion about Jacobson pairs in R, as follows.

Corollary 4.5. *Let R be a strongly π-regular ring. If (α, β) is a Jacobson pair in R, then the spectral idempotents of α and β are similar.*

Proof. From Ara's Theorem 3 in [**1**], regular elements in R are unit-regular, so R is an IC ring by [**15**, Theorem 1.1]). Therefore, the desired conclusion follows from part (E) of Theorem 4.4. □

One of the most famous results in the classical theory of generalized inverses is "Cline's Formula" stated in Theorem 4.6 below, which was proved in [**5**] seven years after the introduction of the theory of Drazin inverses in [**10**]. It took many more years for Jacobson's Desert Island Formula to be extended to the case of Drazin inverses, but still, no link has ever been provided between the two formulas. In the following, we'll show that Cline's Formula is actually a natural consequence

[4]The proof of [**3**, Theorem 3.6] was harder and more indirect since it involved a lengthy reduction to the case where α is group-invertible.

of the (generalized) Desert Island Formula, by a quick application of the "complementation principle" via the Jacobson map $r \mapsto 1-r$ for rings. We have to be careful here, since in general this map does not preserve Drazin invertibility (or its quasi- and pseudo-versions). However, it *does* preserve strong cleanness, while the first half of Theorem 4.4 is fully applicable to strongly clean decompositions. With this observation, a judicious application of part (A) and part (E) of Theorem 4.4 yields quickly a new conceptual derivation of "Cline's Formula" for polar (or Drazin invertible) elements in the second part of the theorem below.

THEOREM 4.6. *For any $a, b \in R$, the strongly clean decompositions of ab are in one-to-one correspondence with those of ba. If ab is polar with Drazin index n and Drazin inverse z, then ba is polar with Drazin index $\leq n+1$ and Drazin inverse bz^2a.*

PROOF. The first part of the theorem follows from Theorem 3.7 and the fact that the strongly clean decompositions of ab (resp. ba) are in one-to-one correspondence with those of $1-ab$ (resp. $1-ba$). For the second part of the theorem, assume ab is polar, and let $ab = e_0 - x$ be the Fitting decomposition of ab, so $(abe_0)^n = 0$ and ab has Drazin inverse $z = -ex^{-1}$ where $e := 1 - e_0$. Then $\alpha := 1 - ab = e + x$ is a strongly clean decomposition. If $f + y$ is the corresponding strongly clean decomposition of $\beta := 1 - ba$, we have the strongly clean decomposition $ba = (1-f) + (-y)$. We'll show that $[ba\,(1-f)]^{n+1} = 0$. Using the commutation rule $af = ea$, we have $ba\,(1-f) = ba - bea = b\,(1-e)\,a$. Since $[ab\,(1-e)]^n = (abe_0)^n = 0$, it follows that $[ba\,(1-f)]^{n+1} = 0$. This shows that ba is polar, with Drazin index $\leq n+1$. Finally, the Drazin inverse of ba is given by $f(-y)^{-1}$. Using Theorem 4.4(A) and the commutation rule $fb = be$, we have

$$\begin{aligned} f(-y)^{-1} &= -f\,[1 + b\,(x^{-1} - ex^{-2})\,a] = -f - b\,(ex^{-1})\,a + be^2x^{-2}a \\ &= b\,(ex^{-1})^2a = bz^2a, \end{aligned}$$

upon recalling that $f = -b\,(ex^{-1})\,a$. □

With the above proof in place for Cline's Formula for polar elements, we can now quickly get its full analogues for both the quasipolar and the pseudopolar elements, as follows.

THEOREM 4.7. *For any $a, b \in R$, if $ab \in R$ is quasipolar with quasi-Drazin inverse z, then ba is quasipolar with quasi-Drazin inverse bz^2a. If ab is pseudopolar instead, a similar result also holds.*

PROOF. We will only handle the quasipolar case, as the pseudopolar case is nearly identical. In the quasipolar case, the proof of Theorem 4.6 can be carried over essentially verbatim. Here, we would start with $ab = e_0 - x$ being the quasi-Fitting decomposition of ab, so $e_0 \in C^2(ab)$ and abe_0 is quasi-nilpotent. Writing $\alpha := 1 - ab = e + x$ as before where $e := 1 - e_0 \in C^2(ab) = C^2(\alpha)$. Using the same notations as in the proof of Theorem 4.6, we have a corresponding strongly clean decomposition $\beta := 1 - ba = f + y$. Then Theorem 2.5(H) gives $f \in C^2(\beta)$; or equivalently, $1 - f \in C^2(ba)$. Arguing as in the proof of Theorem 4.6 (and recalling the statement $(*)$ in the proof of Proposition 3.3), we see that $abe_0 = ab\,(1-e)$ being quasi-nilpotent implies that $ba\,(1-f)$ is quasi-nilpotent. From the strongly clean decomposition $ba = (1-f) + (-y)$ where $1 - f \in C^2(ba)$, it follows that ba is quasipolar. The formula for the quasi-Drazin inverse of ba now follows from the same calculation as in the last part of the proof of Theorem 4.6. □

To complete our references to the literature, we note that Cline's Formula in the quasipolar case was proved by Liao, Chen and Cui in [**22**, Theorem 2.2], and by Gürgün in [**12**, Theorem 2.8]. The corresponding result for the pseudopolar case was first proved by Wang and Chen in [**26**, Theorem 3.6], and later by Gürgün in [**12**, Theorem 2.9].

Example 4.8. Revisiting the theme of Proposition 2.8, we'll give here an explicit example in which $ab = ba$ (so $\alpha = \beta$), but the correspondence of strongly clean decompositions for $\alpha = \beta$ constructed in Theorem 2.5 is far from being the identity map. Let $R = \mathbb{M}_2(S)$ where S is a nonzero ring in which $1 = u + w$ for two suitable units u, w, and let $a = \begin{pmatrix} u & -u \\ 0 & u \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $ab = ba = u\,I_2$, $\alpha := I_2 - ab = w\,I_2$ forms a Jacobson pair with $\beta = I_2 - ba = \alpha$. For any $s \in S$ such that $su = us$, consider the idempotent matrix $e = \begin{pmatrix} 1 & s \\ 0 & 0 \end{pmatrix}$. For $x := \alpha - e = \begin{pmatrix} -u & -s \\ 0 & w \end{pmatrix} \in \mathrm{U}(R)$, we check easily that $ex = xe$, so we have a strongly clean decomposition $\alpha = e + x$. Since $be = e$ is not equal to $eb = \begin{pmatrix} 1 & 1+s \\ 0 & 0 \end{pmatrix}$, we know from Proposition 2.8 that the decomposition $\alpha = e + x$ does not correspond to itself (in the sense of Theorem 2.5). To compute the "corresponding" strongly clean decomposition $f + y$ for $\beta = \alpha$, recall from Theorem 4.4(A) that f is given by $-bex^{-1}a$. A quick computation shows that $f = \begin{pmatrix} 1 & s-1 \\ 0 & 0 \end{pmatrix} \neq e$, so we have the following two "corresponding" (but distinct) strongly clean decompositions (for any $s \in S$ commuting with u):

$$\alpha = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -u & -s \\ 0 & w \end{pmatrix} = \begin{pmatrix} 1 & s-1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -u & 1-s \\ 0 & w \end{pmatrix}.$$

Of course, there are two other (more obvious) strongly clean decompositions for α too; namely, $\alpha = 0 + \alpha$, and $\alpha = I_2 - uI_2$. Under the correspondence constructed in Theorem 2.5, each of these decompositions corresponds to itself. (After all, these decompositions involve only the trivial idempotents, which are central.) Finally, we observe that all of the constructions in this example work already in the ring of upper-triangular matrices $\mathbb{T}_2(S)$. However, the existence of the equation $1 = u + w$ in S did play an essential role, as it is well known (e.g. from [**4**]) that $\mathbb{T}_n(S)$ is a uniquely strongly clean ring for any $n \geq 1$ if S is a Boolean ring.

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