SYMMETRIES IN IDEMPOTENT FACTORIZATIONS

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Abstract. In analogy to the fact that isomorphic idempotents arise from flipped factorizations, we show that under some weak conditions, partial factorizations of the complement idempotents may also be preserved. Applying these results, we give new characterizations of exchange rings. We also show that the definition of an exchange element is left-right symmetric.

1. Motivation and Conventions

Throughout this paper, R is a unital, possibly non-commutative ring. We are primarily interested in the set $\mathrm{idem}(R)$ of idempotents in R; those elements $e \in R$ such that $e^2 = e$. In a ring, each idempotent e has a corresponding complement idempotent $1-e$, and as is well-known the pair $(e, 1-e)$ corresponds to an internal direct sum decomposition of the free right R-module $R_R = eR \oplus (1-e)R$. (Of course, there is also a corresponding left R-module decomposition as well.) We say that eR is the summand of R_R corresponding to the idempotent e, with complement summand $(1-e)R$.

Two idempotents $e, f \in \mathrm{idem}(R)$ are said to be *isomorphic* if $eR \cong fR$ as right R-module; in other words, their corresponding summands are isomorphic. It is a classical theorem that this notion is left-right symmetric, and can be characterized via a factorization property. Namely, e and f are isomorphic if and only if there exist some $a, b \in R$ such that $e = ab$ and $f = ba$. For a proof of this and other characterizations, as well as additional information about isomorphic idempotents, see [6, Proposition 21.20].

Given $e \in \mathrm{idem}(R)$, it is tempting to think that to every factorization $e = ab$ (for some $a, b \in R$) there is a corresponding isomorphic idempotent ba. Unfortunately, ba need not be idempotent in general. However, this can be quickly remedied, by instead noting that $f := baba$ is an idempotent, since

$$f^2 = b(ab)^3a = be^3a = bea = baba = f.$$

Moreover, e and f are isomorphic via the factorizations $e = a(bab)$ and $f = (bab)a$. Thus, every factorization of an idempotent can be slightly modified, and then yields a corresponding isomorphic idempotent. In this paper, we'll call this the "flip and double trick."

When flipping factorizations of an idempotent e, a natural problem is to analyze how much control we should expect to have on similar factorizations of the complement idempotent $1-e$. It turns out that a great deal of regularity occurs in many natural situations. In Section 2 we will prove a number of such general theorems. Theorem 2.1 gives necessary and sufficient conditions for control to exist in general, while Proposition 2.7 gives some streamlined sufficient conditions to guarantee control of a (right) factor of $1-e$ when flipping a single factor of e. As one application, we prove that idempotents in exchange rings possess quite general factorization properties; see Corollary 2.9.

Our second application of these techniques occurs in Section 3, where we study strengthenings of the notion of *suitable elements*, as defined by Nicholson [7]. (The definition will be given below.) These stronger element-wise conditions are pertinent to the study of lifting idempotents in rings.

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2. Main Theorem

Changing notations slightly, given $e \in \mathrm{idem}(R)$ we will consider general factorizations of the form $e \in aRbp$ for some $a, b, p \in R$. We are seeking a new isomorphic idempotent of the form $f \in paRb$. Thus, the factor p is the *flipped* factor. The reason we have the components a and b is to demonstrate, further, that we have control on the initial and final factors not being flipped. We don't specify e's middle factor (between a and b) because we need some freedom if we are to restrict factorizations of $1-e$ and $1-f$. Speaking of which, we similarly want to control the initial and final factors of $1-e$ and $1-f$, thus we seek to guarantee $1-e, 1-f \in cRd$ for some $c, d \in R$.

In our first theorem, we are able to give weak conditions which completely characterize when these types of factorizations are preserved.

Theorem 2.1. *Let $a, b, c, d, p \in R$, and assume there is an idempotent $e \in aRbp$ such that $1-e \in cRd$. There exists an idempotent $f \in peaRb \subseteq paRb$ such that $1-f \in cRd$ if and only if there exists some element $q \in eRb$ such that $1-pq \in cR \cap Rd$. Moreover, if these equivalent conditions hold, then we can guarantee that e and f are isomorphic.*

Proof. The forward direction is an obvious weakening, so we focus on proving the backwards direction. Write $e = axbp$ for some $x \in R$. The "flip and double trick" from the introduction guarantees that $h := peaxb \in \mathrm{idem}(R)$. Note that $hp = pe$, so $(1-h)pq = p(1-e)q = 0$ as $q \in eR$. Now, using the fact $1-pq \in cR$, we find

$$1 - hpq = (1-h)pq + (1-pq) = 1 - pq \in cR. \tag{2.2}$$

Similarly, the containment $1 - pq \in Rd$ guarantees

$$1 - h = (1-h)(1-pq) + (1-h)pq = (1-h)(1-pq) \in Rd. \tag{2.3}$$

Setting $f := h + hpq(1-h)$ an easy computation shows $f \in \mathrm{idem}(R)$. Also, as $q \in Rb$, we have

$$f = h + hpq - hpqh = h + peq - hpqh \in peaRb.$$

Note that the complementary idempotent to f factors as $1 - f = (1 - hpq)(1-h)$, so by (2.2) and (2.3) it belongs to cRd.

Finally, note that e and h are isomorphic, since h arises from the flip and double trick. On the other hand $f = h(1+pq(1-h))$ and $h = (1+pq(1-h))h$, so f and h are isomorphic. As isomorphism among idempotents is transitive, we have the last statement of the theorem. □

Remark 2.4. The proof of Theorem 2.1 is completely explicit; the idempotent f is given by the exact formula

$$paxbpaxb(1 + pq(1 - paxbpaxb)).$$

Moreover, the fact that e and f are isomorphic can be seen directly, by flipping the initial factor $paxbp$.

In many applications we do not have much control over the idempotent e, and thus we would like to reformulate the conditions on q so as to avoid explicitly mentioning e. This is done by replacing the assumption $q \in eR$ with two "commuting conditions" as follows:

Theorem 2.5. *Let $a, b, c, d, p \in R$ and assume there is an idempotent $e \in aRbp$ such that $1-e \in cRd$. If p commutes with c, and there exists an element $q \in Rb$ such that $1 - pq \in cR \cap Rd$ and q commutes with d, then there exists an idempotent $f \in peaRb \subseteq paRb$ such that $1 - f \in cRd$, and f is isomorphic to e.*

Proof. We follow the proof of Theorem 2.1, letting x, h, and f be given in precisely the same way as before. The only change in assumptions is that we no longer know that $q \in eR$, which was only used to guarantee that $(1-h)pq = 0$. Instead, we will show that $(1-h)pq \in cRd$, which will finish the proof since then the containments expressed in (2.2) and (2.3) will still hold true.

Since p commutes with c and $1-e \in cR$, we have $p(1-e) \in cR$. Similarly, $(1-e)q \in Rd$. Now, as before we have $hp = pe$, and thus

$$(1-h)pq = p(1-e)q = p(1-e)(1-e)q \in cRd$$

as desired. □

Following Nicholson [7], an element $a \in R$ is *suitable* if there exists an idempotent $e \in Ra$ such that $1-e \in R(1-a)$. Nicholson [8] and (in unpublished work) A.L.S. Corner proved that this is a left-right symmetric notion. Recently, two of the authors in joint work with T.Y. Lam proved in [4] a "left-right mixed" version of this property.

Proposition 2.6 ([4, Theorem 3.2(2)]). *The following are equivalent for an element $a \in R$.*

(1) *There exists an idempotent $e \in Ra$ such that $1-e \in R(1-a)$.*
(2) *There exists an idempotent $e \in aR$ such that $1-e \in R(1-a)$.*

With Theorem 2.5 in hand, we can do better.

Proposition 2.7. *Given $s, t \in R$, consider the two conditions:*

(1) *There exists an idempotent $e \in Rs$ such that $1-e \in Rt$.*
(2) *There exists an idempotent $e \in sR$ such that $1-e \in Rt$.*

(A) *We have (2) $\Rightarrow$ (1) if $Rs + Rt = R$ and s commutes with t.*
(B) *We have (1) $\Rightarrow$ (2) if there exists some $q \in R$ such that $1 - sq \in Rt$ and q commutes with t.*

Proof. (A) Assume that $Rs + Rt = R$. We can then write $1 - qs \in Rt$ for some $q \in R$. Taking $p = s$, $a = b = d = 1$, and $c = t$, then (the left-right analog of) the assumptions of Theorem 2.5 boil down to assuming (2) holds and that s and t commute, while the conclusion of the theorem is exactly (1).

(B) Taking $p = s$, $a = b = c = 1$, and $d = t$, then the assumptions of Theorem 2.5 boil down to assuming (1) holds and that q and t commute, while the conclusion of the theorem is precisely (2). □

Remark 2.8. In part (A) of Proposition 2.7, note that the unimodular condition $Rs+Rt = R$ is an easy consequence of condition (1), but does not follow from (2), and thus provides a convenient necessary condition for (1) to hold. However, the commuting condition, $st = ts$, is not strictly necessary. For instance in the ring $R := \mathbb{M}_2(F)$ over any field F, taking $s := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $t := \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ we see that both conditions (1) and (2) hold, although s and t do not commute.

On the other hand, this commuting condition is not superfluous; that is, (1) does not simply follow from (2) and the unimodular condition $Rs+Rt = R$. This can be proved using a universal construction, but the details are somewhat complicated and unenlightening, so we don't include them here.

Similar comments apply to part (B) of the proposition. Thus, the two unimodular conditions in Proposition 2.7 represent fundamental assumptions that must be present in some form, whereas the two commuting conditions are serviceable assumptions that can technically be weakened, but not merely dropped.

Notice that when $t = 1 - s$, both the unimodularity and commuting conditions of Proposition 2.7 hold for free (taking $q = 1$), so Proposition 2.6 follows as a corollary. Even more generally we have:

Corollary 2.9. *Let $m, n \in \mathbb{Z}_{\geq 0}$, and let $a \in R$. The following are equivalent:*

(1) *There exists an idempotent $e \in Ra^m$ such that $1-e \in R(1-a)^n$.*
(2) *If $i, j, k, \ell \in \mathbb{Z}_{\geq 0}$ are such that $i + j = m$ and $k + \ell = n$, then there exists an idempotent $e \in a^i R a^j$ such that $1 - e \in (1-a)^k R(1-a)^\ell$.*

If R is an exchange ring (i.e., every element of R is suitable), then these conditions hold for all $m, n \in \mathbb{Z}_{\geq 0}$ and any $a \in R$.

Proof. Given integers $m, n \geq 0$, there are polynomials $f(x), g(x) \in \mathbb{Z}[x]$ such that

$$a^m f(a) + (1-a)^n g(a) = 1.$$

In particular, note that this is a unimodular equation which lies in the *commutative* subring of R generated by a.

We now need to show that we can freely flip powers of a, or of $1-a$, in factorizations of idempotents as above. By left-right symmetry, and also symmetry in passing between a and $1-a$, it suffices to show that if there exists an idempotent $e \in a^i R a^{m-i}$ such that $1-e \in (1-a)^k R (1-a)^{n-k}$ for some $0 \leq i < m$ and $0 \leq k \leq n$, then there exists an idempotent $f \in a^{i+1} R a^{m-i-1}$ such that $1-f \in (1-a)^k R (1-a)^{n-k}$. This follows by applying Theorem 2.5 with $p = a$, $q = a^{m-1} f(a)$, $a = a^i$, $b = a^{m-i-1}$, $c = (1-a)^k$, and $d = (1-a)^{n-k}$.

For the last sentence, note that when R is an exchange ring then $a^m f(m)$ is suitable, so there exists some idempotent $e \in R a^m f(m) \subseteq R a^m$ such that $1 - e \in R(1 - a^m f(m)) = R(1-a)^n g(a) \subseteq R(1-a)^n$. $\square$

The proof of the last sentence of the previous corollary takes some care, as it does not suffice to simply assume some power of a is suitable in the ring.

A nice special case of the corollary is when $m = n = 1$.

Corollary 2.10. *The following are equivalent:*

1. *There exists an idempotent $e \in Ra^2$ such that $1 - e \in R(1-a)^2$.*
2. *There exists an idempotent $e \in aRa$ such that $1 - e \in (1-a)R(1-a)$.*

Surprisingly, even if a^2 is suitable, that does not imply that there exists an idempotent $e \in aRa$ such that $1 - e \in (1-a)R(1-a)$. The converse is also false, as the next example demonstrates.

Example 2.11. *When $m = n = 1$, the equivalent conditions of the Corollary 2.9 are independent of a^2 being suitable.*

Construction. Let $R := \mathbb{M}_2(\mathbb{Z})$ and $a := \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$. We compute that $a^2 = \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix}$ is suitable as

$$e := \begin{pmatrix} -3 & -1 \\ 12 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \in a^2 R$$

is an idempotent, and

$$1 - e = \begin{pmatrix} 4 & 1 \\ -12 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix} \in (1 - a^2)R.$$

However $(1-a)^2 = \begin{pmatrix} 0 & -3 \\ 0 & 9 \end{pmatrix} \in 3R$, so we see that $(1-a)^2 R$ contains only the zero idempotent. As $a^2 R \neq R$, there is no idempotent $f \in a^2 R$ such that $1 - f \in (1-a)^2 R$.

To show independence in the other direction, let $R := \mathbb{Z}$ and take $a := 2$. It is easy to verify that a^2 is not suitable in R, but taking $e := 0 \in Ra^2$ we have $1 - e = 1 \in \mathbb{Z} = R(1-a)^2$. $\square$

There is another natural situation where the existence of complementary idempotents is easy to demonstrate. Recall the well-known fact that a left ideal Rp is a direct summand of ${}_R R$ if and only if p is von Neumann regular, meaning there exists some $p' \in R$ such that $pp'p = p$. The criterion for the existence of a complementary idempotent in cRd in this case is quite minimal.

Proposition 2.12. *Let $p, c, d \in R$ with p von Neumann regular. There exist an idempotent $e \in Rp$ such that $1 - e \in cRd$ if and only if $1 \in Rp + cRd$.*

Proof. The forward direction is clear, without the regularity assumption on p.

Conversely, suppose $1 = xp + cyd$ for some $x, y \in R$. Also write $pp'p = p$ for some $p' \in R$. The element $f := p'p \in Rp$ is an idempotent. Moreover

$$1 - f = (xp + cyd)(1 - p'p) = x(p - pp'p) + cyd(1 - f) = cyd(1 - f) \in cR.$$

Thus, by the flip and double trick, $1 - e := (1 - f)cyd(1 - f)cyd = (1 - f)cyd$ is an idempotent, and it belongs to cRd. Moreover,

$$e = 1 - (1 - f)cyd = (1 - cyd) + fcyd = xp + f(1 - xp) \in Rp$$

as desired. □

This proposition can be thought of as a variant of Nicholson's useful result on summands in projective modules [7, Lemma 2.8].

3. Left-right Symmetry of Exchange Elements

Exchange rings were first defined by Warfield [9] in terms of a direct sum decomposition property on finitely generated free modules. Subsequently, Nicholson [7] showed these rings are exactly those for which idempotents lift modulo all ideals. Moreover, his notion of *suitability*, which we defined in the previous section, captures at an element-wise level much of the information needed to lift idempotents.

As has been long known, there is a hierarchy of lifting hypotheses on ideals; see [5] for a small sampling of such hypotheses. In this section, we study related element-wise lifting conditions which appear in the literature. We begin with a definition found in [2]; an element $a \in R$ is called *left exchange* if for each $b \in R$ with $Ra + Rb = R$, there exists an idempotent $e \in Ra$ such that $1 - e \in Rb$. These elements form a natural generalization of suitable elements.

Following [5], a left ideal I of a ring R is said to be *fully lifting* if idempotents lift to R modulo every left ideal J contained in I. (These were just called *lifting* right ideals in [3].) Based on this notion, let us say that $a \in R$ is a *left fully lifting* element if Ra is fully lifting as a left ideal of R.

Similarly, we will call $a \in R$ a *left fully suitable* element if every element of Ra is suitable in R. Such elements have also been named exchange elements in the literature, but we avoid this terminology for obvious reasons.

Our first result describes the natural implications between these properties.

Proposition 3.1. *For an element $a \in R$ consider the following conditions:*

(1) *a is left fully lifting in R.*
(2) *a is left fully suitable in R.*
(3) *a is left exchange in R.*
(4) *a is suitable in R.*

We have $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$*, but none of the implications is reversible in general.*

Proof. $(1) \Rightarrow (2)$: This is a consequence of [5, Lemma 5.8], which says that if a left ideal I of R is fully lifting, and $x \in R$ is such that $x - x^2 \in I$, then x is suitable in R. In particular, any $x \in I$ is automatically suitable in R.

$(2) \Rightarrow (3)$: Suppose $xa + yb = 1$ for some $b, x, y \in R$. As xa is suitable, there exists an idempotent $e \in Rxa \subseteq Ra$ such that $1 - e \in R(1 - xa) = Ryb \subseteq Rb$.

$(3) \Rightarrow (4)$: This is clear because $Ra + R(1 - a) = R$.

$(4) \not\Rightarrow (3)$: The element $2 \in \mathbb{Z}$ is suitable, but 2 is not left exchange because $2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$ but there does not exist an idempotent $e \in 2\mathbb{Z}$ such that $1 - e \in 3\mathbb{Z}$. Therefore 2 is not left exchange in $\mathbb{Z}$.

(3) $\not\Rightarrow$ (2): Let $R := \mathbb{M}_2(\mathbb{Z})$, and $a := \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$. As a is von Neumann regular, it is left exchange by [7, Lemma 2.8]. However, $Ra^2 = 2R$ and so the only idempotent in Ra^2 is zero, and $R(1-a^2) \neq R$, so a is not left fully suitable.

(2) $\not\Rightarrow$ (1): Every element of the Jacobson radical of any ring is suitable, but idempotents may not lift modulo the Jacobson radical. For instance, if $R \subsetneq \mathbb{Q}$ is the set of elements whose 2-adic and 3-adic valuations are non-negative, then idempotents do not lift modulo $J(R)$. □

Nicholson [8] proved that suitability is left-right symmetric. We will now prove similarly that left exchange elements are right exchange, and vice versa. To prove this, it is convenient to state an alternate characterization of left exchange elements.

Lemma 3.2. *An element $a \in R$ is left exchange if and only if for every $r \in R$ there exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - ra)$.*

Proof. The forward direction is a tautological weakening. For the converse, suppose $Ra + Rb = R$ for some $b \in R$. In particular, we can write $xa + yb = 1$ for some $x, y \in R$. By the hypothesis, there exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - xa) = Ryb \subseteq Rb$. Hence a is left exchange. □

Lemma 3.3. *For $a, r \in R$ the following conditions are equivalent:*

(1) *There exists an idempotent $e \in Ra$ such that $1 - e \in R(1 - ra)$.*
(2) *There exists an idempotent $e \in aR$ such that $1 - e \in R(1 - ar)$.*
(3) *There exists an idempotent $e \in aR$ such that $1 - e \in (1 - ar)R$.*
(4) *There exists an idempotent $e \in Ra$ such that $1 - e \in (1 - ra)R$.*

Proof. (1) ⇒ (2): Let $e = xa$ and $1 - e = y(1 - ra)$ for some $x, y \in R$. From the flip and double trick, $f := axax \in \mathrm{idem}(R) \cap aR$. Moreover, since $fa = ae$, we find

$$\begin{aligned} 1 - f &= (1-f)(1-ar) + (1-f)ar = (1-f)(1-ar) + a(1-e)r \\ &= (1-f)(1-ar) + ay(1-ra)r = (1-f)(1-ar) + ayr(1-ar) \in R(1-ar). \end{aligned}$$

(2) ⇒ (3): Let $e = ax$ and $1 - e = y(1 - ar)$ for some $x, y \in R$. From the flip and double trick $g := (1 - ar)y(1 - ar)y \in \mathrm{idem}(R) \cap (1 - ar)R$. Setting $f := g + g(1 - ar)(1 - g)$ an easy computation shows that $f \in \mathrm{idem}(R) \cap (1 - ar)R$. Moreover,

$$1 - f = (1 - g(1 - ar))(1 - g)$$

and

$$1 - g(1 - ar) = 1 - (1 - ar)(1 - e) = 1 - (1 - e) + ar(1 - e) = e + ar(1 - e) \in aR.$$

Therefore $1 - f$ satisfies the needed conditions for (3).

(3) ⇒ (4) ⇒ (1): These follow by symmetry. □

In view of Lemmas 3.2 and 3.3, we have:

Theorem 3.4. *Every left exchange element is right exchange, and vice versa.*

It turns out that fully suitable elements are also left-right symmetric. One argument is as follows. First, it was proved in [5, Theorem 5.10] that if I is a fully lifting left ideal of R, then I is a (possibly non-unital) exchange ring. The same proof works more generally when I is a left ideal consisting of suitable elements of R. Thus, if $a \in R$ is left fully suitable, then Ra is a (possibly non-unital) exchange ring, and conversely (thus giving an alternate characterization of left fully suitable elements). By [1, Proposition 1.3], the right ideal aR is also an exchange ring, and in particular a is right fully suitable.

The two results cited above are quite involved, but the left-right symmetry of fully suitable elements can also be quickly deduced using Theorem 3.4 as shown below. (Hereafter, we will again only work with unital rings, although the proof we give easily adapts to the non-unital case as well.)

Corollary 3.5. *Each left fully suitable element is right fully suitable, and vice versa.*

Proof. To ease notation, let $\operatorname{suit}(R)$ denote the set of suitable elements of a ring R. Assume $a \in R$ is left fully suitable. Given any $r \in R$, we will show that $ar \in \operatorname{suit}(R)$.

By our assumption, and Proposition 3.1, ara is a left exchange element. Hence, by Theorem 3.4, ara is a right exchange element as well. As $araR+(1-ar)R = R$, there exists an idempotent $e \in araR \subseteq arR$ such that $1 - e \in (1 - ar)R$. Hence $ar \in \operatorname{suit}(R)$. $\square$

It is not always the case that left-right symmetry holds for similar conditions. For instance, if every element of Ra is von Neumann regular, then it need not be the case that the same is true of aR. To see this, take F to be a field and $R := \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, and let $a := E_{11}$. Now, every element of Ra is regular, but $E_{12} \in aR$ is not regular in R.

We end with the following related question, which we were unable to answer:

Question 3.6. Are left fully lifting elements also right fully lifting?

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