SYMBOLIC EXTENSIONS FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

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Abstract. We show there are no symbolic extensions C^1-generically among diffeomorphisms containing nonhyperbolic robustly transitive sets with a center indecomposable bundle of dimension at least 2. Similarly, C^1-generically homoclinic classes with a center indecomposable bundle of dimension at least 2 that satisfy a technical assumption called index adaptation have no symbolic extensions.

1. Introduction

Symbolic dynamical systems arose as a tool to code complicated dynamics by the use of more tractable systems (shift spaces). The existence of Markov partitions for hyperbolic systems provides an effective coding of these systems using symbolic dynamics. A natural question that we address in the present work is to what extent a nonhyperbolic system can be codified using symbolic systems.

A dynamical system (X, f) has a *symbolic extension* if there exists a subshift (Y, σ) and a continuous surjective map $\pi : Y \to X$ such that $\pi \circ \sigma = f \circ \pi$; the system (Y, σ) is called an *extension* of (X, f) and (X, f) is called a *factor* of (Y, σ).

The results in this paper are a first step in proving the following general principle.

General Principle. *Diffeomorphisms with a dominated splitting (for the precise definition see Section 2.3) of the form $E^s \oplus E^c \oplus E^u$ such that E^s is uniformly contracting, E^u is uniformly expanding (these bundles*

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may be trivial), and E^c is a nonhyperbolic central bundle that splits in a dominated way into 1-dimensional subbundles have a symbolic extension, while those with center (nonhyperbolic) indecomposable bundles of dimension at least 2, C^1 generically, do not have a symbolic extension.

This general principle is based on some previous partial results obtained by several authors, for example [2, 23, 33]. In the present work we address the existence of symbolic extensions for diffeomorphisms with a 1-dimensional center bundle. In a forthcoming work the authors together with M. J. Pacifico, and J. Vieitez will address the case where the center bundle splits into 1-dimensional subbundles. We now state the precise definitions and theorems related to the general principle.

1.1. **Existence of symbolic extensions.** Every dynamical system with a symbolic extension has finite entropy. J. Auslander asked if the converse holds: namely, whether every system with finite entropy has a symbolic extension. In fact, the existence of symbolic extensions is related to the notion of asymptotically h-expansive systems, as defined by Misiurewicz [30] (in Section 2 we review all relevant notions in the introduction). A nice form of a symbolic extension is a *principal extension*, that is an extension given by a factor map which preserves entropy for every invariant measure. Boyle, D. Fiebig, and U. Fiebig [12] were able to show the following:

Theorem 1.1 ([12]). *If (X, f) is asymptotically h-expansive, then f has a principal symbolic extension.*

Using a result of Buzzi [16], that states that any C^∞ diffeomorphism is asymptotically h-expansive one gets the following:

Corollary 1.2 ([12, 16]). *Every C^∞ map of a compact manifold has a principal symbolic extension.*

This result shows the connections between the regularity of the maps and the existence of symbolic extensions.

We let $\mathrm{Diff}^r(M)$ be the space of C^r diffeomorphisms of a closed compact manifold M endowed with the usual uniform topology. When $r = 1$ we simply write $\mathrm{Diff}(M)$.

Conjecture 1.3 (Downarowicz and Newhouse [23]). *If $f \in \mathrm{Diff}^r(M)$ and $r \geq 2$, then f has a symbolic extension.*

In a recent work Downarowicz and Maass [22] have shown the above conjecture for maps of a closed interval. See also the paper by Burguet proving that C^2-surface diffeomorphisms have symbolic extensions [14].

Besides the regularity of the system a second ingredient for the existence of symbolic extensions seems to be hyperbolic-like properties. This is precisely the idea behind our results. We now give a description of h-expansive systems also called entropy expansive systems, for a precise definition see Section 2. A diffeomorphism is *h-expansive* if for any sufficiently small $\epsilon > 0$ and any $x \in M$ the set of points whose forward orbit stays ϵ-close to the forward orbit of x has topological entropy zero.

Theorem 1.4 (Cowieson and Young, Proposition 6 in [20])**.** *Every partially hyperbolic C^1-diffeomorphism with a 1-dimensional center bundle is h-expansive (entropy expansive) and therefore has a principal symbolic extension.*

We prove the above result in Section 3 as the goal of this paper is to address the role of partial hyperbolicity in the setting of symbolic extensions. The proof is essentially the same as in [20].

The above result supports the principle of Pugh and Shub that "a little hyperbolicity goes a long way" [35] and the suggestion made to us by Burns [15] that 1-dimensional center partially hyperbolic systems in many ways behave like the hyperbolic ones. For instance, for these systems the so called entropy conjecture by Shub ([37]) holds: the entropy of f is greater than or equal to the spectral radious of the map induced by f in the real homology group, see [36].

We note that Pacifico and Vieitez [33] proved that a homoclinic class that is robustly isolated and robustly h-expansive has a partially hyperbolic splitting where the center direction splits into 1-dimensional center directions. In another paper Pacifico and Vieitez [34] gave some conditions that guarantee a system is h-expansive.

From the above theorem we obtain a result concerning existence of equilibrium states for partially hyperbolic diffeomorphisms with 1-dimensional center bundle. Before stating the corollary we define equilibrium states. For X a compact metric space and f a homeomorphism of X we let $\mathcal{M}(f)$ be the set of invariant Borel probability measures for f. If f is a homeomorphism of a compact metric space and $\varphi \in C^0(X)$, then the *pressure of f with respect to φ and $\mu \in \mathcal{M}(f)$* is $P_\mu(\varphi, f) := h_\mu(f) + \int \varphi d\mu$. The concept of topological pressure, denoted $P(\varphi, f)$, for a system is a generalization of topological entropy and corresponds to a "weighted" topological entropy, see [27, p. 623]. In fact, the topological pressure reduces to the topological entropy when $\varphi = 0$. The *variational principle* for pressure states that if X is a compact metric space, f is a homeomorphism of X, and $\varphi \in C^0(X)$, then

$$P(\varphi, f) := \sup_{\mu \in \mathcal{M}(f)} P_\mu(\varphi, f).$$

A measure μ such that $P(\varphi, f) = P_\mu(\varphi, f)$ is called an *equilibrium state* and corresponds to a natural generalization of a measure of maximal entropy. When $\varphi = 0$ the topological pressure is the topological entropy and an equilibrium state is a measure of maximal entropy. The next result is a standard consequence of entropy-expansivity.

Corollary 1.5. *If $V \in C^0(M)$ and $f \in \mathrm{Diff}(M)$ is partially hyperbolic with 1-dimensional center bundle, then there is an equilibrium state for (M, V). In particular, f has a measure of maximal entropy.*

We consider two closely related classes of invariant sets. Namely, robustly transitive sets and homoclinic classes. We now give the precise definitions. Given an open set U and a diffeomorphism f, we let $\Lambda_f(U) := \bigcap_{n \in \mathbb{Z}} f^n(\bar{U})$. We say that the set $\Lambda_f(U)$ is *robustly transitive* if $\Lambda_f(U) \subset U$ and for every g that is C^1 close to f the set $\Lambda_g(U)$ is *transitive* (has a point with a forward dense orbit). When U is the whole manifold we say f is *robustly transitive.*

The first examples of robustly transitive sets were the hyperbolic basic sets (including transitive Anosov diffeomorphisms). Examples of nonhyperbolic robustly transitive diffeomorphisms were constructed by Shub [38], Mañé [29], and others [3, 10]. Finally, the robustly transitive sets were introduced in [21] as a generalization of these examples.

Theorem 1.6 ([8, 21])**.** *Every robustly transitive set has a dominated splitting.*

In fact, there is a "finest" dominated splitting: that is a dominated splitting $E_1 \oplus \cdots \oplus E_{k(f)}$ of $T_{\Lambda_f(U)}M$ such that each subbundle E_i of the splitting is *indecomposable* (i.e. there is no dominated splitting $G_i \oplus F_i$ of E_i), see [8, Theorem 4]. Additionally, some of the examples above are partially hyperbolic and have a 1-dimensional center bundle.

Homoclinic classes were introduced by Newhouse [31] as a generalization of the hyperbolic basic sets. For a hyperbolic periodic point p of a diffeomorphism f its *homoclinic class, $H(p, f)$*, is the closure of the transverse intersections of the stable and unstable manifolds associated with the orbit of p. In contrast to the robustly transitive sets, homoclinic classes may fail to have any kind of weak hyperbolicity. For instance, they may not have a dominated splitting (the simplest example is a homoclinic class on a surface with a homoclinic tangency).

If a homoclinic class has a dominated splitting, then the finest dominated splitting for a homoclinic class is defined as above. We have the following result that is an immediate consequence of Theorem 1.4.

Corollary 1.7. *Every robustly transitive set or homoclinic class whose center bundle is 1-dimensional has a principal symbolic extension.*

As a remark, we observe that this result shows that super-exponential growth of the number of periodic points does not preclude the existence of symbolic extensions. Specifically, the C^1-generic nonhyperbolic diffeomorphisms satisfying Corollary 1.7 have super-exponential growth of the number of periodic points [7].

1.2. **Non-existence of symbolic extensions.** Boyle, D. Fiebig, and U. Fiebig [12] constructed a number of topological examples that have no symbolic extensions. The first examples of diffeomorphisms with no symbolic extensions were constructed by Downarowicz and Newhouse [23]. They show that a generic area-preserving C^1 diffeomorphism of a surface is either Anosov or has no symbolic extension. This result relies on the existence of persistent homoclinic tangencies.

Asaoka [2] (see also [40]) gives examples of diffeomorphisms in a 3-dimensional disk with C^1 persistent homoclinic tangencies, and uses the result of Downarowicz and Newhouse, to show that for any smooth manifold M with $\dim(M) \geq 3$, there exists an open subset of $\mathrm{Diff}(M)$ in which generic diffeomorphisms have no symbolic extensions.

We now describe another class of C^1 generic diffeomorphisms without symbolic extensions, and show how this is related with non-hyperbolic properties.

1.2.1. *Robustly transitive sets.* Let U be an open set in a closed manifold M. Let $\mathcal{T}(U)$ be the open set of diffeomorphisms, f, such that $\Lambda_f(U)$ is robustly transitive. Denote by $\mathcal{T}^{nh}(U)$ the subset of $\mathcal{T}(U)$ of diffeomorphisms, f, such that $\Lambda_g(U)$ is not hyperbolic for every diffeomorphism g in a C^1 neighborhood of f. This implies that there is an open and dense subset $\mathcal{T}_0^{nh}(U)$ of $\mathcal{T}^{nh}(U)$ of diffeomorphisms g such that $\Lambda_g(U)$ contains saddles having different *s-indices* (dimension of the stable bundle of the saddle). This assertion follows using standard arguments in the C^1-topology. Just observe that in this case there are periodic points in $\Lambda_f(U)$ having arbitrarily large period and whose rate of contraction/expansion is close to one. Then it is possible to change the index of these points after an arbitrarily small perturbation along their orbits. The robustly transitivity assumption implies that these new saddles also belong to the set. See the arguments in [8, 21].

Consider the finest dominated splitting defined over $\Lambda_f(U)$,

$$T_{\Lambda_f(U)}M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$$

where $\dim(E_i) = d_i$ for $1 \leq i \leq k$. We say that $\Lambda_f(U)$ has an *indecomposable central bundle of dimension at least 2* if there are $1 \leq i < k$ and saddles p' and q' in $\Lambda_f(U)$ such that

$$s\text{-index of } p' < d_1 + \cdots + d_i \leq s\text{-index of } q' \leq d_1 + \cdots + d_{i+1}$$

and $d_{i+1} \geq 2$. This means that E_{i+1} is an indecomposable center (non-hyperbolic) bundle of dimension at least 2 since E_{i+1} is contracting for p' and expanding for q'.

Theorem 1. *Given any open set U of M there is a C^1 residual set $\mathcal{R}$ of the open set $\mathcal{T}(U)_0^{nh}$ with the following property.*

Consider a diffeomorphism $f \in \mathcal{T}(U)_0^{nh}$ and a neighborhood $\mathcal{V}_f$ of f such that for every $g \in \mathcal{V}_f \cap \mathcal{R}$ the set $\Lambda_g(U)$ has an indecomposable central bundle of dimension at least 2.

Then for every $g \in \mathcal{R} \cap \mathcal{V}_f$ the set $\Lambda_g(U)$ has no symbolic extension. In particular, any $g \in \mathcal{R} \cap \mathcal{V}_f$ has no symbolic extension.

The proof of this this theorem has two main ingredients: First, the existence of an indecomposable central bundle of dimension two (or greater) yields homoclinic tangencies associated to saddles of $\Lambda_f(U)$. Second, since $f \in \mathcal{T}(U)_0^{nh}$ the set $\Lambda_f(U)$ contains saddles of different indices, this allows these tangencies to persist.

Relevant systems satisfying Theorem 1 are the DA-diffeomorphisms of the 3-torus,$\mathbb{T}^3$, obtained via Hopf bifurcations having a central non-hyperbolic bundle of dimension two. In this case $\mathbb{T}^3 = U = \Lambda_f(U)$ and f is partially hyperbolic. Another example of the same nature are the diffeomorphisms of the four torus, $\mathbb{T}^4$, that are DA-like diffeomorphisms. In this case, U is the entire manifold $\mathbb{T}^4$ and the unique non-trivial dominated splitting of the system is of the form $E \oplus F$ where both E and F are two-dimensional and nonhyperbolic. For details see [10].

1.2.2. *Homoclinic classes.* There is a corresponding result of Theorem 1 for homoclinic classes. As in the case of robustly transitive sets, given a saddle p of a diffeomorphism f there is defined the finest dominated splitting of its homoclinic class $H(p, f)$

$$T_{H(p,f)}M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$$

where $\dim(E_i) = d_i$ for $1 \leq i \leq k$ and each bundle is indecomposable. The finest dominated splitting is uniquely defined.

In principle, a nonhyperbolic homoclinic class $H(p, f)$ may contain saddles having s-indices different from the one of p (in fact, in the case of nonhyperbolic robustly transitive sets this is a typical feature). We say that the homoclinic class $H(p, f)$ has an *indecomposable central bundle of dimension at least 2 adapted to its s-indices* [1] if there are $1 \leq i < k$ and saddles p' and q' in $H(p, f)$ such that

$$s\text{-index of } p' < d_1 + \cdots + d_i \leq s\text{-index of } q' \leq d_1 + \cdots + d_{i+1}.$$

As above, the indecomposable bundle E_{i+1} is nonhyperbolic and its dimension is at least 2. In the next theorem the fact that there is a non-dominated central bundle of dimension greater than or equal to 2 is a key ingredient. As in Theorem 1, this property allows us to get homoclinic tangencies. The fact that the indecomposable central bundle is adapted to the s-indices of the homoclinic class allows us to make these tangencies persistent.

Theorem 2. *There is a residual set $\mathcal{R}$ of* $\mathrm{Diff}(M)$ *with the following property. Consider $f \in \mathcal{R}$, a saddle p_f of f, and a neighborhood $\mathcal{V}_f$ of f in* $\mathrm{Diff}(M)$ *where the continuation p_g of p_f is defined for all $g \in \mathcal{V}_f$.*

Suppose that for every $g \in \mathcal{V}_f \cap \mathcal{R}$ the homoclinic class $H(p_g, g)$ has an indecomposable central bundle of dimension at least 2 adapted to its s-indices.

Then for every $g \in \mathcal{R} \cap \mathcal{V}_f$ the homoclinic class $H(p_g, g)$ does not admit a symbolic extension. In particular, any $g \in \mathcal{R} \cap \mathcal{V}_f$ has no symbolic extension.

An equivalent formulation of Theorem 2 is the following. For each $k \in \mathbb{N}$, consider the subset $\mathrm{Per}(H(p_f, f))_k$ of $H(p_f, f)$ consisting of saddles of s-index k (this set may be empty) and the stable/unstable splitting $E^s \oplus E^u$ defined over the saddles in $\mathrm{Per}(H(p_f, f))_k$.

Theorem 3. *There is a residual $\mathcal{R}$ of* $\mathrm{Diff}(M)$ *with the following property. Consider $f \in \mathcal{R}$, a saddle p_f of f, and a neighborhood $\mathcal{V}_f$ of f in* $\mathrm{Diff}(M)$ *where the continuation p_g of p_f is defined for all $g \in \mathcal{V}_f$.*

Suppose that for every $g \in \mathcal{V}_f \cap \mathcal{R}$ there is k_g such that the stable/unstable splitting defined over $\mathrm{Per}(H(p_f, f))_{k_g}$ *is not dominated.*

Then for every $g \in \mathcal{R} \cap \mathcal{V}_f$ the homoclinic class $H(p_g, g)$ does not admit a symbolic extension. In particular, any $g \in \mathcal{R} \cap \mathcal{V}_f$ has no symbolic extension.

[1] In principle, for homoclinic classes the indecomposable bundle E_1 (for instance) of a C^1 generic diffeomorphism may be nonhyperbolic and every saddle of the homoclinic class may have s-index strictly greater that d_1, this is the reason we speak of a splitting adapted to the s-indices.

We observe that the generic systems satisfying Theorem 1 also satisfy Theorems 2 and 3. Other examples of systems satisfying Theorems 2 and 3 are the diffeomorphisms with persistent tangencies in [2, 4, 6, 40].

The proofs of Theorems 1, 2, and 3 rely on the arguments of Downarowicz and Newhouse in [23]. The idea of the argument is that when there are persistent flat homoclinic tangencies, then it is possible to create arbitrarily small horseshoes with relatively large entropy (*horseshoes with emerging entropy*, see Section 4.1.3). These arguments are all two dimensional and their translation to higher dimensions is not straightforward. As in Asaoka's paper, we borrow these ideas and adapt them to a higher dimensional context. As the proof of the non-existence of symbolic extensions in [2] is quite brief and refers to the arguments in [23] and it considers a very specific dynamical configuration, we consider it is important to give a complete explanation of these argument in the general case, see Section 4.

Let us observe that the main difference between the proofs of Theorem 1 (for robustly transitive sets) and Theorems 2 and 3 (for homoclinic classes) is that in the robustly transitive case the horseshoes with emerging entropy are inside the robustly transitive set by definition, while in the case of homoclinic classes one needs some C^1-generic arguments to guarantee that these horseshoes are in the homoclinic class. This makes the proof for robustly transitive sets simpler.

2. BACKGROUND

In this section we review some facts on subshifts, entropy, asymptotic h-expansivity, hyperbolicity, and weak forms of hyperbolicity.

2.1. **Subshifts.** If we let $\mathcal{A} = \{0, ..., m-1\}$, then the *full m-shift* is the space $\Sigma_m = \mathcal{A}^{\mathbb{Z}}$ endowed with the product topology together with the map $\sigma : \Sigma_m \to \Sigma_m$ defined by $\sigma(s) = t$ where $t_i = s_{i+1}$ for all $i \in \mathbb{Z}$ and all $s = (s_i) \in \Sigma_m$. A *shift space* is a closed, shift invariant subset of a full shift.

2.2. **Entropy.** We now review some basic definitions of topological and measure theoretical entropy, for details see [41]. Let (X, d) be a compact metric space and f be a continuous self-map of X. The d_n metric on X is defined as

$$d_n(x, y) := \max_{0 \le i \le n-1} d(f^i(x), f^i(y))$$

and is equivalent to d and defined for all $n \geq 0$. For a set $Y \subset X$ a set $A \subset Y$ is (n, ϵ)-*spanning* if for any $y \in Y$ there exists a point

$x \in A$ where $d_n(x,y) < \epsilon$. The minimum cardinality of the (n,ϵ)-spanning sets of Y is denoted $r_n(Y,\epsilon)$. The *topological entropy* for a system (X,f) is

$$h_{\text{top}}(f) := \lim_{\epsilon\to 0}\Big(\limsup_{n\to\infty}\frac{1}{n}\log r_n(X,\epsilon)\Big).$$

Let $(X,\mathcal{B},\mu)$ be a Lebesgue measure space with $\mu(X)=1$. A *partition* of X is a collection ξ of measurable sets, $\xi=\{C_\alpha\in\mathcal{B}\,|\,\alpha\in I\}$, such that

- $\mu(X/\bigcup_{\alpha\in I}C_\alpha)=0$,
- $\mu(C_\alpha)>0$ for all $\alpha\in I$, and
- $\mu(C_{\alpha_1}\cap C_{\alpha_2})=0$ for $\alpha_1\neq\alpha_2$.

For partitions ξ and ν the *joint partition of* ξ *and* ν is

$$\xi\vee\nu := \{C\cap D\,|\,C\in\xi,\ D\in\nu,\ \mu(C\cap D)>0\}.$$

Let f be a measure preserving transformation of $(X,\mathcal{B},\mu)$. For a measurable partition ξ and $n\in\mathbb{N}$ we define the *joint partition of* ξ *with respect to* f *for* n to be the partition

$$\xi_n^f=\xi\vee f^{-1}(\xi)\vee\cdots\vee f^{-n+1}(\xi). \tag{1}$$

The *metric entropy of* f *relative to the partition* ξ is

$$h_\mu(f,\xi) := \lim_{n\to\infty}\frac{1}{n}H_\mu(\xi_n^f) \tag{2}$$

where

$$H_\mu(\xi) := -\sum_{\alpha\in I}\mu(C_\alpha)\log\mu(C_\alpha).$$

Then the *entropy of* f *with respect to* μ is

$$h_\mu(f) := \sup\{h_\mu(f,\xi)\,|\,\xi \text{ is a measurable partition with } H_\mu(\xi)<\infty\}.$$

We now review the definitions of h-expansivity and asymptotic h-expansivity. Given a subset $Y\subset X$ we let

$$\bar{r}(Y,\epsilon) := \limsup_{n\to\infty}\frac{1}{n}\log r_n(Y,\epsilon)\quad\text{and}\quad \tilde{h}(f,Y) := \lim_{\epsilon\to 0}\bar{r}(Y,\epsilon).$$

We denote the closed ball with center at x and radius ϵ in the d_n metric as $B_\epsilon^n(x)$. Let

$$\Phi_\epsilon(x) := \bigcap_{n=1}^{\infty}B_\epsilon^n(x).$$

Then

$$h_f^*(\epsilon) := \sup_{x\in X}\tilde{h}(f,\Phi_\epsilon(x)).$$

The map f is *h-expansive* (entropy expansive) if there exists some $c>0$ such that $h^*_f(\epsilon)=0$ for all $\epsilon\in(0,c)$. The map f is *asymptotically h-expansive* if

$$\lim_{\epsilon\to 0} h^*_f(\epsilon)=0.$$

2.3. **Hyperbolicities.** We now review some definitions and facts of weak forms of hyperbolicity, for details we refer to Appendix B.1 in [9]. For a diffeomorphism f of M, an f-invariant set Λ (not necessarily closed) has a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E\oplus F$ such that the bundles E and F are both non-trivial and the fibers $E(x)$ and $F(x)$ have dimensions independent of $x\in\Lambda$ and there exists $\ell\in\mathbb{N}$ such that, for every $x\in\Lambda$ and every pair of unitary vectors $u\in E(x)$ and $v\in F(x)$, it holds that

$$\frac{|Df^\ell_x(u)|}{|Df^\ell_x(v)|}\le\frac{1}{2}.$$

In this case, we write $E\oplus_< F$ to denote that F dominates E.

A Df-invariant bundle F defined over Λ is *indecomposable* if there is no dominated splitting $F=F_1\oplus_< F_2$ over Λ with two non-trivial subbundles.

In some cases, we need to consider splittings having several bundles. A Df-invariant splitting

$$T_\Lambda M=E_1\oplus\cdots\oplus E_k$$

is dominated if for all $i=1,\dots,(k-1)$ the splitting $T_\Lambda M=E_1^i\oplus E_{i+1}^k$ is dominated, where $E_j^\ell=E_j\oplus\cdots\oplus E_\ell$, for $1\le j\le \ell\le k$. In this case we use the notation $E_1\oplus_<\cdots\oplus_< E_k$.

We say that the dominated splitting $T_\Lambda M=E_1\oplus_<\cdots\oplus_< E_k$ of Λ is the *finest dominated splitting* over Λ if for all $i=1,\dots k$ the bundle E_i is *indecomposable.* The finest dominated splitting of a compact f-invariant set is uniquely defined.

An f-invariant compact set Λ is *partially hyperbolic* if it has a dominated splitting

$$T_\Lambda M=E_1\oplus_<\cdots\oplus_< E_k$$

such that either the bundle E_1 is uniformly contracting or the bundle E_k is uniformly expanding. A diffeomorphism f is *partially hyperbolic* if the whole manifold M is partially hyperbolic.

Proposition 2.1 (Appendix B.1 in [9]). *Properties of dominated splittings:*

(1) Continuity and extension to the closure. *A dominated splitting $E_1\oplus_<\cdots\oplus_< E_k$ is continuous. This implies that a dominated*

splitting defined over a set Λ can be extended in a dominated way to the closure of Λ.

(2) Uniqueness. *Consider two f-invariant sets Σ and Δ, $\Delta \subset \Sigma$. Suppose that Σ has a dominated splitting $E_1 \oplus_< \cdots \oplus_< E_k$ and that Δ has a Df-invariant splitting $F_1 \oplus \cdots \oplus F_k$ with $E_i \cap F_i \neq \{\bar{0}\}$ and $\dim(E_i(x)) = \dim(F_i(x))$ for all $x \in \Delta$. Then $E_i(x) = F_i(x)$ for all x and therefore the splitting $F_1 \oplus \cdots \oplus F_k$ over Δ is dominated.*

(3) Persistence. *Dominated splitting and partial hyperbolicity persist under C^1 perturbations. More precisely, consider an f-invariant compact set Λ with a dominated (resp. partially hyperbolic) splitting. Then there are a neighborhood U of Λ and a C^1 neighborhood $\mathcal{V}_f$ of f such that, for all $g \in \mathcal{V}_f$, the maximal invariant set of g in U, $\cap_{n\in\mathbb{Z}} g^n(U)$, has a dominated splitting of the same type as the one of Λ.*

An application of Proposition 2.1 is the following: Consider dominated splittings defined over infinite sets of hyperbolic periodic points. Then the closure of this set of hyperbolic periodic points has an extension of the splitting and the splitting is dominated.

For a partially hyperbolic diffeomorphism f with a splitting $E^s \oplus E^c \oplus E^u$ there exist unique families $\mathcal{F}^u$ and $\mathcal{F}^s$ of injectively immersed submanifolds such that $\mathcal{F}^i(x)$ is tangent to E^i for $i = s, u$, and the families are invariant under f. These are called, respectively, the unstable and stable foliations of f. For the center direction it is known, in the general case, that there is no foliation tangent to the center bundle [24]. If the center bundle is 1-dimensional, then there always exist curves tangent to the center direction through every point. Such curves are called *central curves*. However, these curves need not be unique due to the fact that the center bundle, in general, is not Lipschitz.

3. Partially hyperbolic diffeomorphisms with 1-dimensional center foliations

In this section we outline the proof of Theorem 1.4 and prove Corollaries 1.5 and 1.7. We will see that the set of points in $\Phi_\epsilon(x)$ (for ϵ sufficiently small) will be a subset of the ϵ-center-stable set for x. In the stable direction there is no exponential growth in the spanning sets and the existence of a 1-dimensional center bundle implies that the growth of spanning sets is at most subexponential. This implies that for ϵ sufficiently small and any $x \in M$ we will have $\tilde{h}(f, \Phi_\epsilon(x)) = 0$. So f is h-expansive.

3.1. **Proof of Theorem 1.4.** Before proceeding we define some notation. For a curve η in M its length is denoted by $|\eta|$.

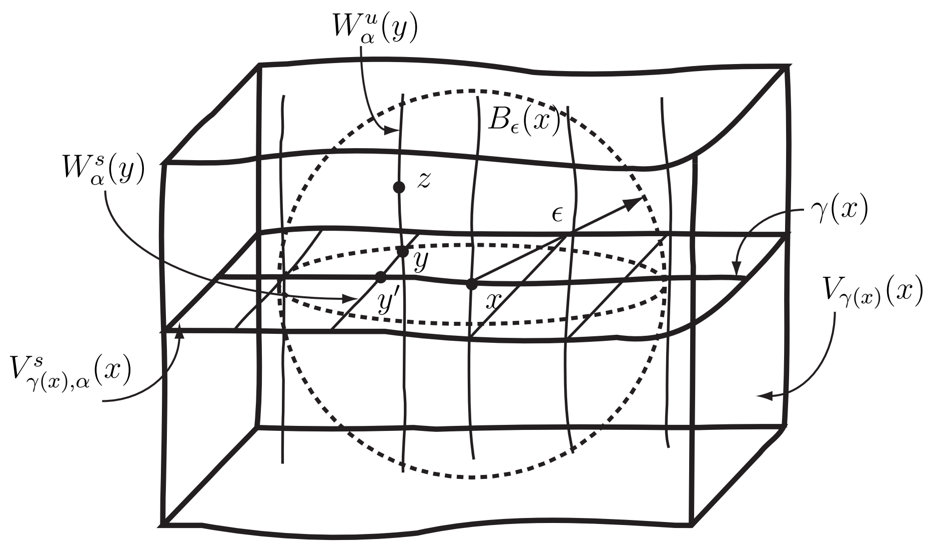

FIGURE 1. Foliation box

Let $f \in \mathrm{Diff}(M)$ be partially hyperbolic with 1-dimensional center bundle. Throughout the rest of the proof we assume that $TM = E^s \oplus E^c \oplus E^u$ with each subbundle nontrivial. The case where E^s or E^u is trivial is a simplification and is omitted. We let $\dim(E^s) = s$.

The first step of the proof is to show that there exists a "foliation chart" on a uniform scale. More precisely, define λ as the maximal expansion of the derivative of f and select constants $\delta_0, \delta_1, \delta_2 > 0$ satisfying the following:

- for every curve γ tangent to E^c with $|\gamma| < \lambda\,\delta_0$ and every s-disk $\Delta^s \subset \mathcal{F}^s$ with s-diameter less than δ_1 one has that $\gamma \cap \Delta^s$ contains at most one point; and
- for every $(s+1)$-disk Υ of the form
$$\Upsilon = \bigcup_{y\in\gamma} \mathcal{F}^s_{\delta_1}(y),$$
where γ is a curve tangent to E^c with length bounded by δ_0 and $\mathcal{F}^s_{\delta_1}(y)$ is the δ_1-disk centered at y contained in $\mathcal{F}^s$, and every u-disk $\Delta^u \subset \mathcal{F}^u$ with u-diameter less than δ_2 one has that $\Upsilon \cap \Delta^u$ contains at most one point.

We select small $\delta > 0$ such that $\lambda\,\delta < \delta_i$ for $i = 0, 1, 2$.

Fix $\alpha_0 \in (0, \delta/2)$. For each $x \in M$ consider a curve $\gamma(x)$ tangent to E^c containing x with radius α_0. (So the extreme points of $\gamma(x)$ are at distance α_0 from x, where $\alpha_0 < \delta/2$.) This is possible since E^c is 1-dimensional. By construction, $\mathcal{F}^s_{\delta_1}(z) \cap \gamma(x)$ consists of at most one point for all x and z. We let

$$V^s_{\gamma(x),\alpha_0}(x) := \bigcup_{y\in\gamma(x)} \mathcal{F}^s_{\alpha_0}(y).$$

For small $\tau, \alpha \in (0, \alpha_0)$ we define

$$V^{s,u}_{\gamma(x),\alpha,\tau}(x) := \bigcup_{z\in V^s_{\gamma(x),\alpha(x)}} \mathcal{F}^u_\tau(z).$$

Finally, for notational simplicity, we let $\alpha = \tau$ and write

$$V_{\gamma(x)}(x) := V^{s,u}_{\gamma(x),\alpha,\alpha}(x).$$

Note that every $V_{\gamma(x)}(x)$ contains a ball $B_\epsilon(x)(x)$, where $\epsilon(x)$ is a lower semi-continuous function. Therefore, by compacity of M, for every $\epsilon > 0$ sufficiently small and $x \in M$ we have

$$B_\epsilon(x) \subset V_{\gamma(x)}(x).$$

Note that this inclusion holds for all $\gamma(x)$ as above. See Figure 1.

Remark 3.1 (Foliation chart)**.** *As in* [20] *we have selected* δ_0 *in such a way that for all central curve* η *of size* $\lambda\,\delta_0$ *the cube*

$$V_\eta = V^{s,u}_{\eta,\alpha,\alpha}$$

provides a foliation chart. (This means that locally the strong stable and strong unstable leaves are diffeomorphic to E^s *and* E^u*. See for instance* [18, p. 19] *for a precise definition of a foliation chart.)*

Lemma 3.2. *For every* $x \in M$ *and* ϵ *small enough it holds that*

$$\Phi_\epsilon(x) = \bigcap_{n=1}^{\infty} B^n_\epsilon(x) \subset V^s_{\gamma(x),\alpha}(x).$$

Proof: Consider the forward orbit of x and define $x_n = f^n(x)$. For each n we define a central curve $\gamma_n = \gamma(f^n(x)) = \gamma(x_n)$ such that γ_n is "compatible" with $f(\gamma_{n-1})$. Meaning that each component of $f(\gamma_{n-1}) - \{f^n(x)\}$ either contains a component of $\gamma_n - \{f^n(x)\}$ (so the component of $f(\gamma_{n-1}) - \{f^n(x)\}$ overflows a component of $\gamma_n - \{f^n(x)\}$) or the component is a subset of one of the components of $\gamma_n - \{f^n(x)\}$.

Define the components $\gamma^\pm_n(x_n)$ of $\gamma_n(x_n) \setminus \{x_n\}$, then the sets $f(\gamma^\pm_n)$ and $\gamma^\pm_{n+1}$ are either nested or disjoint.

The "cubes" $V_{\gamma_n}(x_n)$ containing $B_\epsilon(x_n)$ are defined as above.

Let $z \in \Phi_\epsilon(x)$. By definition and construction, $f^n(z) = z_n \in V_{\gamma_n}(x_n)$ for all $n \geq 0$. We let $y = y(z)$ be the unique point in $V^s_{\gamma(x),\alpha}(x)$ such that $z \in \mathcal{F}^u_\alpha(y)$ and $y' = y'(z)$ be the unique point in $\gamma(x)$ such that $y \in \mathcal{F}^s_\alpha(y')$. See Figure 1.

Write $y_n = y(z_n)$ and $y'_n = y'(z_n)$.By construction, $y'_n \in \gamma_n$, otherwise, by Remark 3.1, $z_n \notin B_\epsilon(x_n)$.

If $z_n \neq y_n$, then the uniform expansion along the unstable direction implies that there is a first n such that z_n and y_n can not be in the same foliation chart of radius α. To be more precise, by hypothesis, we know that y_n is in $V^s_{\gamma_n}(x_n)$ for all n. If $z_n \neq y_n$ for some n the uniform expansion along the unstable direction implies that there is a first m such that $z_m \notin V_{\gamma_m}(x_m)$. Hence, $z_m \notin B_\epsilon(x_m)$, a contradiction. This ends the proof of the lemma. □

Define

$$\Gamma^c(x) := \bigcap_{n\geq 0} f^{-n}(\gamma_n).$$

Given a central curve η of size at most δ_0 and a point $y \in V^s_{\eta,\alpha}$ we let y' be the unique point in η such that $y \in \mathcal{F}^s_\alpha(y')$. The definition of y and y' immediately gives the following:

Fact 3.3. *Let $y \in \Phi_\epsilon(x)$. Then $y' \in \Gamma^c(x)$.*

This fact implies that

$$|f^n(\Gamma^c(x))| < 2\,\delta_0.$$

A folklore fact implies that the growth of an (n,δ)-spanning set in $\Gamma^c(x)$ is subexponential for all $\delta > 0$, see for instance [17, Lemma 3.2].

The next lemma will show that f is h-expansive, completing the proof of Theorem 1.4.

Lemma 3.4. *For every $x \in M$ and all $\delta > 0$ small enough,*

$$\tilde{r}(\Phi_\epsilon(x),\delta) = \limsup_{n\to\infty} \frac{1}{n} r_n(\Phi_\epsilon(x),\delta) = 0.$$

Proof: Since f is uniformly contracting in the stable direction we know that any exponential growth of the spanning sets for $\Phi_\epsilon(x)$ occurs along $\Gamma^c(x)$. This means that we can focus on the growth of (n,δ)-spanning set in $\Gamma^c(x)$. More precisely:

Fact 3.5. *Let Y be an $(n,\delta/2)$-spanning set in $\Gamma^c(x)$. Then Y is a (n,δ)-spanning set in $V^s_{\Gamma^c(x),\alpha}$ for all n sufficiently large.*

By the comments above, this growth is subexponential and the proof of the theorem is complete. □

3.2. **Proof of Corollary 1.5.** The existence of a principal symbolic extension implies the existence of equilibrium states. This result appears to be folklore so we provide a proof for completeness.

Indeed, assume there is a principal symbolic extension, (Y, σ), of (X, f) with a semi-conjugacy π. So if μ is an invariant Borel probability measure for (Y, σ), then the measure theoretic entropy of μ is equal to the measure theoretic entropy of $\pi_*(\mu)$, i.e., $h_\mu = h_{\pi_*(\mu)}$; see [3]. Consider a potential $V \in C^0(X)$. Then $V \circ \pi$ is a continuous potential on Y. Let μ be a invariant Borel probability measure for (Y, σ). Then $\mu \to \mu(V)$ is continuous (where the topology on the set of invariant Borel probability measures is the weak* topology). As Y is expansive we know that the function mapping μ to the measure theoretic entropy with respect to μ is upper semi-continuous; see for instance [32]. Hence, the pressure on $(Y, V \circ \pi)$ is upper semi-continuous and $(Y, V \circ \pi)$ has at least one equilibrium m; see for instance [28, Theorem 4.2.3]. Since (Y, σ) is a principal symbolic extension we know that $\pi_*(m)$ is an equilibrium state of (X, V).

3.3. **Proof of Corollary 1.7.** It is enough to argue exactly as in Subsection 3.1 considering a small neighborhood of the robustly transitive set where the bundles can be extended (recall Proposition 2.1) and consider balls in this neighborhood.

4. Diffeomorphisms with no symbolic extensions

This section consists of two parts. We first consider in Section 4.1 robustly transitive sets and prove Theorem 1 about non-existence of symbolic extensions for a large class of these sets (those having an indecomposable central bundle of dimension at least two).

The arguments for proving analogous results for homoclinic classes (Theorems 2 and 3) are similar, but they involve some additional technical difficulties, see Section 4.2. This is the reason we first discuss the simple case of robustly transitive sets. This allows us to present the main ideas in the proofs while avoiding technicalities.

A necessary condition for a system to have no symbolic extension is that there is additional entropy that is "hidden" in the multi-scale structure of the dynamics. For the systems we investigate, with no symbolic extensions, we will have the following principle in [23]:

General Principle: Obstruction for the existence of symbolic extensions: *there exists some $\epsilon > 0$ such that at any η-scale (where $\eta > 0$) there exists a horseshoe with entropy at least ϵ that at the η-scale appears to be a periodic orbit with zero entropy.*

These horseshoes we call *horseshoes with emerging entropy.* More precise definitions are given in Section 4.1.3.

A first step in the construction of such systems is the existence of a central indecomposable bundle of dimension at least two that yields persistence of tangencies. After a small perturbation these tangencies generate horseshoes with emerging entropy inside the transitive set. This fact will prevent the existence of symbolic extensions.

In fact, all periodic points and horseshoes (including those with emerging entropy) obtained throughout our construction are in an arbitrarily small neighborhood of the transitive set. Thus, by definition of robustly transitive set, they are inside the transitive sets. In the case of homoclinic classes, one needs to guarantee that the horseshoes with emerging entropy are contained in the class. This relies on properties of C^1 generic diffeomorphisms and a careful analysis of the generation of these horseshoes.

4.1. **Robustly transitive sets.** In this section we first explain how the existence of a center indecomposable bundle of dimension at least 2 generates persistent homoclinic tangencies, see Proposition 4.2. This proposition is proved in Section 4.1.2. In Section 4.1.3, to get the non-existence of symbolic extensions, we will use the 2-dimensional arguments in [23]. For that we need to consider special saddles in the robustly transitive set (saddles with real multipliers) and see that the tangencies occur in a local normally hyperbolic surface. These tangencies yield horseshoes with emerging entropy. We explain how the non-existence of symbolic extensions follows from the existence of such horseshoes. Finally, in Section 4.1.4, we prove Theorem 1.

4.1.1. *Persistence of homoclinic tangencies.* Let us first introduce some notations and review the constructions in [4, 8, 11].

Consider an open set U of M. Recall that $\mathcal{T}(U)$ is the (open) set of diffeomorphisms f such that $\Lambda_f(U)$ is robustly transitive and that $\mathcal{T}^{nh}(U)$ is the (open) subset of $\mathcal{T}(U)$ of diffeomorphisms f such that the set $\Lambda_g(U)$ is not hyperbolic for every diffeomorphism g close to f. We now let $\mathcal{T}_2^{nh}(U)$ be the open subset of $\mathcal{T}^{nh}(U)$ of diffeomorphisms f such that there is a neighborhood $\mathcal{V}$ of f such that for each $g \in \mathcal{V}$ the set $\Lambda_g(U)$ has a nonhyperbolic center indecomposable bundle of dimension at least 2.

Define $\mathcal{N}(U)$ as the set of diffeomorphisms $f \in \mathrm{Diff}(M)$ such that $\Lambda_f(U)$ is robustly transitive and contains periodic points of different s-indices (dimension of the stable bundle). By definition $\mathcal{N}(U) \subset$

$\mathcal{T}^{nh}(U)$. We let $s^-(f,U)$ and $s^+(f,U)$ be the minimum and maximum of the indices of periodic points of $\Lambda_f(U)$. The set $\mathcal{N}(U)$ is open and dense in $\mathcal{T}^{nh}(U)$, see [11].

Given a diffeomorphism $f \in \mathcal{N}(U)$, we consider the finest dominated splitting $E_1 \oplus_< E_2 \oplus_< \cdots \oplus_< E_{k(f,U)}$ of $\Lambda_f(U)$. We let $\alpha = \alpha(f,U)$ be the first $j \in \{1,\ldots,k(f,U)\}$ such that E_j is not uniformly contracting. Similarly, we let $\beta = \beta(f,U)$ be the last $j \in \{1,\ldots,k(f,U)\}$ such that E_j is not uniformly expanding. Then $E_1 \oplus \cdots \oplus E_{\alpha-1}$ is uniformly contracting and $E_{\beta+1} \oplus \cdots \oplus E_{k(f,U)}$ is uniformly expanding.

Proposition 4.1 (Corollary C in [11])**.** *There is an open and dense subset $\mathcal{V}(U)$ of $\mathcal{N}(U)$ such that following maps are locally constant: the number $k(f,U)$ of bundles of the finest dominated splitting, the dimensions d_i of these bundles, and the numbers $\alpha(f,U)$ and $\beta(f,U)$. Furthermore, the functions $s^-(f,U) = s^-$ and $s^+(f,U) = s^+$ are locally constant in $\mathcal{V}(U)$ and for every $j \in [s^-, s^+]$ the set of saddles of s-index j is dense in $\Lambda_f(U)$.*

We are ready to state a key result about persistence of tangencies.

Proposition 4.2. *Let $\mathcal{V}(U)$ be the open and dense subset of $\mathcal{N}(U)$ in Proposition 4.1. Consider $f \in \mathcal{V}(U) \cap \mathcal{T}_2^{nh}(U)$ and $l \in [\alpha(f,U), \beta(f,U)]$ such that $d_l \geq 2$. Suppose that there is a saddle $p_f \in \Lambda_f(U)$ and $k \geq 1$ such that*

$$d_1 + \cdots + d_{l-1} + k \leq s\text{-}index(p_f) < d_1 + \cdots + d_{l-1} + d_l$$

whose continuation is defined for every g in a C^1 neighborhood $\mathcal{U}_f \subset \mathcal{V}(U) \cap \mathcal{T}_2^{nh}(U)$ of f. Then there is a dense subset $\mathcal{D}$ of $\mathcal{U}_f$ consisting of diffeomorphisms with homoclinic tangencies associated to p_g.

We observe that this proposition is a particular case of Proposition 4.17 (see [25, Theorem 6.1.1]) and it admits a quite simple proof that we will sketch for completeness.

Remark 4.3. *If $f \in \mathcal{V}(U) \cap \mathcal{T}_2^{nh}(U)$ then by Proposition 4.1 there exists some $l \in [\alpha, \beta]$ such that $d_l \geq 2$ and therefore there is a saddle p_f as in Proposition 4.2.*

Before proceeding with the proof of Proposition 4.2 we define some notation and make some comments. Fix a component $\mathcal{C}(U)$ of $\mathcal{V}(U) \cap \mathcal{T}_2^{nh}(U)$ where the maps in Proposition 4.1 are constants and let

$$r = d_1 + \cdots + d_{l-1} + k. \tag{3}$$

We consider the following subsets of $\Lambda_f(U)$:

- $\mathrm{Per}(f,U)$ is the set of periodic points $q \in \Lambda_f(U)$,

- $\operatorname{Per}(f,U)_j$ is the subset of $\operatorname{Per}(f,U)$ of periodic points of s-index j.

Given a periodic point q we denote by $\tau(q)$ its period and by

$$\lambda_1(q),\dots,\lambda_n(q)$$

the eigenvalues of $Df^{\tau(q)}(q)$ counted with multiplicity and ordered in non-decreasing modulus. We say that $\lambda_i(q)$ is the i^{th} *eigenvalue* of q.

- $\operatorname{Per}_{\mathbb{R}}(f,U)$ is the subset of $\operatorname{Per}(f,U)$ of periodic points q whose eigenvalues $\lambda_i(q)$ are all real and different in modulus.
- $\operatorname{Per}_{\mathbb{R}}(f,U)_j := \operatorname{Per}_{\mathbb{R}}(f,U)\cap\operatorname{Per}(f,U)_j$.
- $\operatorname{Per}_{\mathbb{R}}(f,U)_j^n$ is the subset of $\operatorname{Per}_{\mathbb{R}}(f,U)_j$ of points of period n.
- $\operatorname{Per}_{\mathbb{C}}(f,U)_{j,j+1}$ is the set of periodic points $q\in\Lambda_f(U)$ such that
$$|\lambda_{j-1}(q)|<|\lambda_j(q)|=|\lambda_{j+1}(q)|<1<|\lambda_{j+2}(q)|$$
and $\lambda_j(q)$ and $\lambda_{j+1}(q)$ are non-real. Note that q has s-index $j+1$.

Fix r as in equation (3) and consider the following subsets of $\mathcal{C}(U)$:

- $\mathcal{D}(U) := \{g\in\mathcal{C}(U)\colon\ \operatorname{Per}_{\mathbb{R}}(g,U)_r$ and $\operatorname{Per}_{\mathbb{R}}(g,U)_{r+1}$ are both dense in $\Lambda_g(U)\}$.
- $\mathcal{O}(U) := \{g\in\mathcal{C}(U)\colon$ the sets $\operatorname{Per}_{\mathbb{R}}(g,U)_r$, $\operatorname{Per}_{\mathbb{R}}(g,U)_{r+1}$ and $\operatorname{Per}_{\mathbb{C}}(g,U)_{r,r+1}$ are all non-empty$\}$.

The next remark is just a dynamical reformulation of results about cocycles in [8, Lemmas 4.16 and 5.4], for details see [11].

Remark 4.4. *Let $\mathcal{C}(U)$ and r be as above.*

- *$\mathcal{D}(U)$ is residual in $\mathcal{C}(U)$.*
- *$\mathcal{O}(U)$ is open and dense in $\mathcal{C}(U)$.*

An immediate and standard consequence of Hayashi's Connecting Lemma [26] is the next remark.

Remark 4.5. *There is an open and dense subset of $\mathcal{C}(U)$ such that for every $g\in\mathcal{C}(U)$ and any pair of saddles $q_g\in\operatorname{Per}(f,U)_{r+1}$ and $q_g'\in\operatorname{Per}(f,U)_r$ one has that $W^s(q_g)$ and $W^u(q_g')$ have some transverse intersection. Moreover, there is h arbitrarily close to g such that $W^u(q_h)\cap W^s(q_h')\neq\emptyset$ (i.e., the diffeomorphism h has a heterodimensional cycle associated with q_h and q_h').*

4.1.2. *Sketch of the proof of Proposition 4.2.* If we omit the condition of the persistence of the tangencies, this proposition is just Theorem F in [11]. We now explain how the persistence is obtained. This follows from standard arguments in the C^1 topology. We recall the main steps.

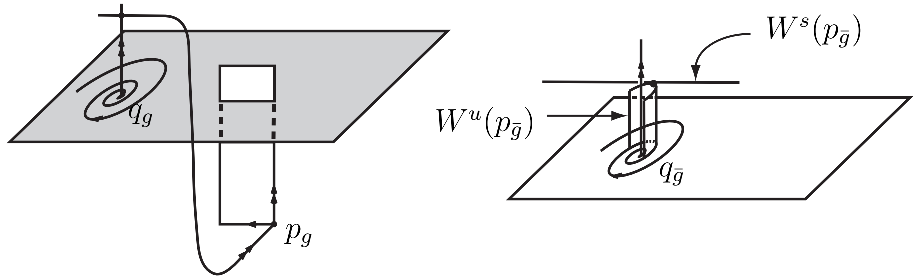

FIGURE 2. Unfolding of a cycle for p_g to create a tangency

Fix a saddle p_g of s-index r as in the proposition. By Remark 4.4, after a perturbation, we can assume that $\Lambda_g(U)$ contains a saddle $q_g \in \mathrm{Per}_{\mathbb{C}}(g,U)_{r,r+1}$. Applying Remark 4.5 with $q'_g = p_g$, we can assume that $W^u(p_g)$ and $W^s(q_g)$ has some transverse intersection. Thus, we can assume that $W^u(p_g)$ spiralizes around $W^u(q_g)$. By Remark 4.5, after a perturbation, we can get an intersection between $W^s(p_g)$ and $W^u(q_g)$ (obtaining a cycle associated to p_g and q_g). Unfolding this cycle, one gets a homoclinic tangency associated to p_g. These arguments are depicted in Figure 2. See for instance Section 2.2 of [4] for details.

4.1.3. *Downarowicz-Newhouse construction.* To begin this section we review the condition given in [23] that guarantees the non-existence of symbolic extensions (see Theorem 4.7). Using Proposition 4.2 we will then verify that there is a residual set $\mathcal{R}$ in $\mathcal{T}_2^{nh}(U)$ such that each $f \in \mathcal{R}$ satisfies this condition, and therefore every $f \in \mathcal{R}$ has no symbolic extension.

As we will examine invariant measures for a system (X, f), we review some basic facts of invariant measures. Let $\mathcal{M}(f, X)$ be the set of invariant probability measures and $\mathcal{M}_e(f, X)$ be the set of ergodic invariant probability measures for (X, f). If f is a homeomorphism of a compact metric space, then $\mathcal{M}(f, X)$ is nonempty and satisfies the following (see [13, Section 4.6]):

- $\mathcal{M}(f, X)$ is convex and compact for the weak*-topology, and
- $\mathcal{M}_e(f, X)$ is precisely the extreme points of $\mathcal{M}(f, X)$.

Let $\rho(\cdot, \cdot)$ denote a metric on $\mathcal{M}(f, X)$ giving the weak*-topology.

A sequence of partitions $\{\alpha_k\}$ is *essential* if

- $\text{diam}(\alpha_k) \to 0$ as $k \to 0$, and
- $\mu(\partial\alpha_k) = 0$ for any $\mu \in \mathcal{M}(f, X)$, where $\partial\alpha_k$ is the union of the boundaries of the elements in the partition α_k.

A *simplicial sequence of partitions* is a sequence $\{\alpha_k\}$ of nested partitions (each α_k is a refinement of α_{k-1}) whose diameters go to zero and such that for each α_k the partition is given by a smooth triangulation of M.

Proposition 4.6 (Proposition 4.1 in [23]). *Let $\{\alpha_k\}$ be a simplicial sequence of partitions of M. Then there is a residual set $\mathcal{S} \subset \text{Diff}(M)$ such that if $f \in \mathcal{S}$, then $\{\alpha_k\}$ is an essential partition for f.*

The next result is the condition used to prove that there is no symbolic extension.

Theorem 4.7 (Proposition 4.4 in [23]). *Let $\{\alpha_k\}$ be an essential sequence of partitions on M. Suppose there exists an $\epsilon > 0$ and a compact set $\mathcal{E} \subset \mathcal{M}(f, M)$ such that for every $\mu \in \mathcal{E}$ and every $k > 0$,*

$$\limsup_{\nu \to \mu, \nu \in \mathcal{E}} \big(h_\nu(f) - h_\nu(\alpha_k, f)\big) > \epsilon.$$

Then a symbolic extension for f does not exist.

The construction of the set $\mathcal{E}$ involves measures supported on hyperbolic basic sets. By [39] these measures can be obtained as limits of periodic measures defined as follows: Let $f \in \text{Diff}(M)$ and p be a hyperbolic periodic point for f with period $\tau(p)$. Then the *periodic measure for p* is

$$\mu_p := \frac{1}{\tau(p)} \sum_{i=0}^{\tau(p)-1} \delta_{f^i(p)},$$

where $\delta_{f^i(p)}$ is the point mass at $f^i(p)$.

Using Proposition 4.2 we will construct $\mathcal{E}$ as the closure of a certain set of periodic measures.

Note that it is enough to prove Theorem 1 for the set $\mathcal{V}(U) \cap \mathcal{U}_f$, see Proposition 4.1 and Proposition 4.2. Therefore, the number $r = d_1 + \cdots + d_{l-1} + k$ in Proposition 4.2 remains fixed from now on.

Let $\lambda_j(q)$ be the j-th eigenvalue of a saddle q. We let $\chi_j(q)$ be the j-th Lyapunov exponent of q, i.e.,

$$\chi_j(q) := \frac{1}{\tau(q)} \log(|\lambda_j(q)|),$$

and define

$$\chi_j(f, U) := \inf\{\chi_j(q) \,:\, q \in \text{Per}_{\mathbb{R}}(f, U)_j\}. \tag{4}$$

For an ergodic measure μ we let $\chi_j(\mu)$ be its j-th Lyapunov exponent.

A hyperbolic set Λ is *subordinate* to a partition ξ if there exist compact sets $\Lambda_1, ..., \Lambda_j$ for some $j \geq 1$ such that

- $\Lambda = \bigcup_{i=1}^{j} \Lambda_i$,
- $f(\Lambda_i) = \Lambda_{i+1}$, for $1 \leq i \leq j-1$, and $f(\Lambda_j) = \Lambda_1$, and
- Λ_i is contained in a single element of the partition ξ for all $1 \leq i \leq j$.

Definition 4.8. A diffeomorphism f with an essential sequence of simplicial partitions $\{\alpha_k\}$ satisfies property $\mathcal{S}_j^n$ if for each $q \in \mathrm{Per}_{\mathbb{R}}(f,U)_j^n$ the following hold:

(a) there exists a 0-dimensional hyperbolic basic set $\Lambda(q,n)$ for f such that

$$\Lambda(q,n) \cap \partial\alpha_n = \emptyset \text{ and } \Lambda(q,n) \subset \Lambda_f(U),$$

(b) the set $\Lambda(q,n)$ is subordinate to α_n,

(c) there is a $\mu \in \mathcal{M}_e(f, \Lambda(q,n))$ such that

$$|h_\mu(f) - \chi_j(q)| < \frac{1}{n}\chi_j(q),$$

and

(d) for every $\mu \in \mathcal{M}_e(f, \Lambda(q,n))$ we have

$$\rho(\mu, \mu_q) < \frac{1}{n} \text{ and } |\chi_j(\mu) - \chi_j(q)| < \frac{1}{n}\chi_j(q).$$

Let $f \in \mathcal{O}(U)$ (see Remark 4.4) and let $\tau_r(f,U)$ be the minimum period of points in $\mathrm{Per}_{\mathbb{R}}(f,U)_r$. We let $\mathcal{R}_r^m$ be the set of diffeomorphisms in $\mathcal{O}(U)$ with $\tau_r(f,U) = m$.

For $m \leq n$ let $\mathcal{D}_r^{m,n}$ be the set of $\mathcal{R}_r^m$ of diffeomorphisms satisfying property $\mathcal{S}_r^n$.

Lemma 4.9. *For every m and sufficiently large n the set $\mathcal{D}_r^{m,n}$ is open and dense in $\mathcal{R}_r^m$.*

Proof: The proof of the above lemma follows from the proof of Lemma 5.1 in [23]. Since the sets $\Lambda(q,n)$ are hyperbolic the sets $\mathcal{D}_r^{m,n}$ are open (see [23, page 471]). Thus the crucial point is density and for that we use Proposition 4.2. We now go into the details of the proof.

Fix n and m, with $m \leq n$, and consider a diffeomorphism $g \in \mathcal{R}_r^m$ and a saddle $q \in \mathrm{Per}_{\mathbb{R}}(g,U)_r^n$. By Proposition 4.2, after an arbitrarily small perturbation we can assume that g has a homoclinic tangency associated to q. Note that the r-th and $(r+1)$-th multipliers of q satisfy the following

$$|\lambda_{r-1}(q)| < |\lambda_r(q)| < 1 < |\lambda_{r+1}(q)| < |\lambda_{r+2}(q)|.$$

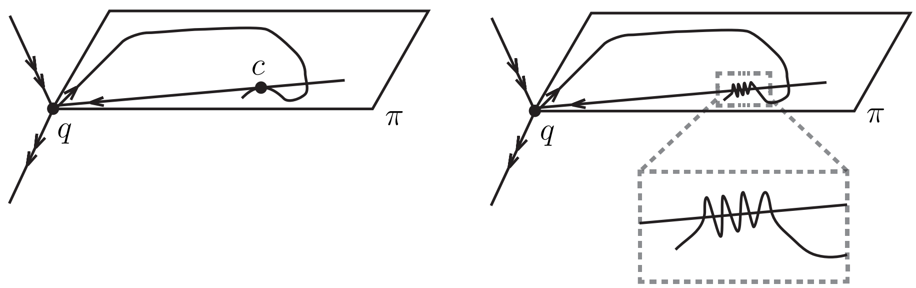

FIGURE 3. The creation of $\Lambda(q,n)$ in the planar region π

Thus, after a new perturbation, we can assume that this tangency occurs in a "normally hyperbolic surface" π (tangent to the eigenspaces corresponding to the $r^{\rm th}$ and $(r+1)^{\rm th}$ eigenvalues of q): so there are curves $\gamma^s \subset W^s(q)\cap\pi$ and $\gamma^u \subset W^u(q)\cap\pi$ with a quadratic homoclinic tangency. See for instance [4, Affirmation] and Figure 3. Thus, from now on the arguments are "2-dimensional" and we can follow [23].

More precisely, let c be the tangency point. After a perturbation, for each large n, we get a hyperbolic set $\Lambda(q,n)$ of f subordinate to the partition α_n and contained in a small neighborhood of the orbit of c such that $h_{\rm top}(f,\Lambda(q,n))$ is close to $\chi_{r+1}(q)$, see [23, equation (43), page 484]. Since $\Lambda(q,n)$ is a transitive hyperbolic set we know there exists an ergodic measure of maximal entropy μ. Hence, $h_\mu(\Lambda(q,n))$ is close to $\chi_j(q)$. By construction, the set $\Lambda(q,n)$ is contained in $\Lambda_f(U)$. Furthermore, the distance between the ergodic measures of $\Lambda(q,n)$ and the periodic measure of q can be made arbitrarily small and the $(r+1)^{\rm th}$ Lyapunov exponent of μ can be chosen to be arbitrarily close to that for q, see [23]. This completes the sketch of the proof of Lemma 4.9. □

Remark 4.10.

(1) *The hyperbolic sets* $\Lambda(q,n)$ *are the horseshoes with emerging entropy described at the beginning of Section 4.*

(2) *By construction, each periodic point* $\bar q \in \Lambda(q,n)$ *satisfies Proposition 4.2 and belong to* $\mathrm{Per}_{\mathbb{R}}(f,U)_r$. *Therefore, we can apply the previous construction to these saddles.*

4.1.4. *Proof of Theorem 1.* Note that it is enough to prove a local version of this theorem for diffeomorphisms $g \in \mathcal{U}_f \cap \mathcal{V}_f \cap \mathcal{S}$, where $\mathcal{S}$ is the residual set in Proposition 4.6. Recall that for these diffeomorphisms the functions α, β, s^-, and s^+ are constant.

The proof is similar to the proof of Theorem 1.3 in [23, pages 471-2]. Recalling equation (4) define

$$\rho_0 := \frac{\chi_{r+1}(g,U)}{2} \tag{5}$$

and let

$$\mathcal{E}_1(g) := \{\mu_q \,:\, q \in \mathrm{Per}_{\mathbb{R}}(g,U)_r \text{ and } \chi_{r+1}(q) > \rho_0\},$$

and set $\mathcal{E}(g)$ as the closure of $\mathcal{E}_1(g)$.

It is sufficient to verify the conditions of Theorem 4.7 for $\mu_q \in \mathcal{E}_1(g)$. More precisely, from Lemma 4.9 we know that for each n sufficiently large there exists a periodic hyperbolic basic set $\Lambda(q,n)$ that is subordinate to α_n and all ergodic measures of $\Lambda(q,n)$ are $\frac{1}{n}$ close to μ_q. Moreover, from (d) in Definition 4.8,

$$\chi_{r+1}(\mu) > \frac{n-1}{n}\,\chi_{r+1}(q)$$

and from (c) in Definition 4.8 for every $\nu_n \in \mathcal{M}_e(g,\Lambda(q,n))$ it holds that

$$h_{\nu_n}(g) > \frac{n-1}{n}\,\chi_{r+1}(q). \tag{6}$$

Lemma 4.11. $\mathcal{M}_e(g,\Lambda(q,n)) \subset \mathcal{E}(g)$.

Proof: Just recall that every ergodic measure supported on a hyperbolic basic set is the weak*-limit of periodic measures for periodic points in the set, see [39]. Moreover, by Remark 4.10, the periodic points in $\Lambda(q,n)$ belong to $\mathrm{Per}_{\mathbb{R}}(g,U)_r$. □

Lemma 4.12. *Let* $\nu_n \in \mathcal{M}_e(g,\Lambda(q,n))$. *Then* $h_{\nu_n}(\alpha_n,g) = 0$.

Proof: Recall that (see equations (1) and (2))

$$h_{\nu_n}(g,\alpha_n) = \lim_{j\to\infty} H_{\nu_n}((\alpha_n)^g_j) = \lim_{j\to\infty} \frac{-1}{j} \sum_{A\in(\alpha_n)^g_j} \nu_n(A)\,\log\nu_n(A).$$

Since $\Lambda(q,n)$ is subordinate to α_n the sum is constant for each j, see [23, 464]. Therefore this limit goes to 0 as $j\to\infty$. □

Fix $k_0 \in \mathbb{N}$ and a measure $\mu \in \mathcal{E}(g)$. By Theorem 4.7 it is enough to prove that there exists a sequence $\nu_n \to \mu$ such that

$$\limsup_{n\to\infty} \big(h_{\nu_n}(g) - h_{\nu_n}(\alpha_{k_0},g)\big) > \rho_0. \tag{7}$$

Fix a sequence $\eta_n \in \mathcal{E}(g)$ converging to μ. By definition of $\mathcal{E}(g)$, we know that for each η_n there is is a periodic measure $\mu_q \in \mathcal{E}_1(g)$ arbitrarily close η_n. Then by (d) in Definition 4.8 and Lemma 4.11 there exists $\nu_n \in \mathcal{M}_e(g, \Lambda(q,n)) \subset \mathcal{E}(g)$ which is $\frac{1}{n}$ close to μ_q.

Take large $n \geq k_0$, the partition α_n is a refinement of α_{k_0}, we have from Lemma 4.12 that $h_{\nu_n}(\alpha_{k_0}, g) = 0$. Then, from (6) and the choice of ρ_0 in equation (5), for n sufficiently large we have

$$h_{\nu_n}(g) - h_{\nu_n}(\alpha_{k_0}, g) = h_{\nu_n}(g) > \frac{n-1}{n}\chi_{r+1}(q) > \frac{\chi_{r+1}(g,U)}{2} = \rho_0.$$

As ν_n converges to μ we get (7), ending the proof of the theorem relative to robustly transitive sets.

To complete the proof of Theorem 1, note that the diffeomorphisms that have no symbolic extension for a robustly transitive set will also have no symbolic extension for the diffeomorphism over the entire manifold. Indeed, if there is a set of invariant measures on $\Lambda_f(U)$ that satisfy Theorem 4.7, then there is a set of invariant measures for f that satisfy Theorem 4.7. □

4.2. **Homoclinic classes.** In this section, we prove Theorems 2 and 3. For that we explain how the constructions in Section 4.1 for robustly transitive sets can be adapted for homoclinic classes having a center indecomposable bundle of dimension at least two. The main difficulty for adapting these arguments is to guarantee that the horseshoes with emerging entropy are contained in the considered homoclinic class.

In the next section, we recall some C^1 generic properties of diffeomorphisms that will guarantee that the horseshoes with emerging entropy can be taken with this property. This will allow to translate the arguments for robustly transitive sets to the setting of homoclinic classes.

4.2.1. *Properties of C^1 generic diffeomorphisms.* There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}(M)$ consisting of diffeomorphisms f satisfying properties (HC1)–(HC5) below:

(HC1) All periodic points of $f \in \mathcal{R}$ are hyperbolic and their stable and unstable manifolds are in general position (Kupka-Smale genericity theorem).

(HC2) Every homoclinic class $H(p_f, f)$ of $f \in \mathcal{R}$ depends continuously on $f \in \mathcal{R}$. Moreover two homoclinic class $H(p_f, f)$ and $H(q_f, f)$ either coincide or are disjoint. In particular, for every saddle $r_f \in H(p_f, f)$ one has $H(r_f, f) = H(p_f, f)$. Moreover, every pair of saddles p and q contained in the same homoclinic class with the same s-index are *homoclinically related* (i.e., the stable manifold of the orbit of p transversely meets the unstable manifold of the orbit of q and vice-versa), see [19, 5].

By item (1) in Proposition 2.1 (continuity of dominated splittings), the first assertion in (HC2) implies that if $f \in \mathcal{R}$ and $H(p_f, f)$ has a dominated splitting $E_1 \oplus_< \cdots \oplus_< E_k$ then, for all $g \in \mathcal{R}$ close to f, the homoclinic class $H(p_g, g)$ of the continuation p_g of p_f also has a dominated splitting $E'_1 \oplus_< \cdots \oplus_< E'_k$ with $\dim(E_i) = \dim(E'_i)$.

(HC3) Suppose that the homoclinic class $H(p_f, f)$, $f \in \mathcal{R}$, contains a saddle q_f. Then there is a neighborhood $\mathcal{V}_f$ of f in Diff(M) such that $H(p_g, g) = H(q_g, g)$ for every diffeomorphism $g \in \mathcal{V}_f \cap \mathcal{R}$. See [1, Lemma 2.1].

Remark 4.13. *The properties above allow us to define the continuation of homoclinic classes H of diffeomorphisms in $f \in \mathcal{R}$. Suppose that p_f is a (hyperbolic) periodic point of f in H. Then $H = H(p_f, f)$. If $g \in \mathcal{R}$ is close enough to f, the continuation H_g of H for g is $H_g = H(p_g, g)$. The continuation H_g does not depend on the choice of the saddle $p_f \in H_f$.*

(HC4) Consider a homoclinic class $H = H(p_f, f)$, $f \in \mathcal{R}$, and let $i^-(H)$ and $i^+(H)$ be the maximum and the minimum of the s-indices of the saddles in H. Then for every $k \in \mathbb{N}$ in the *index interval* $[i^-(H), i^+(H)]$ *of* H there is a saddle q_f of s-index k in H, see [1]. Moreover, by (HC2), $H = H(p_f, f) = H(q_f, f)$.

Property (HC3) implies that for every $g \in \mathcal{R}$ close to f the index interval of H_g contains $[i^-(H), i^+(H)]$ and the maps $g \mapsto i^{\pm}(H_g)$ are semi-continuous. Hence we can take the residual set $\mathcal{R}$ of Diff(M) such that the index interval of any homoclinic class is locally constant.

In what follows, given a homoclinic class H, we denote by $\mathrm{Per}(H)_k$ the subset H of saddles of s-index k.

Remark 4.14. *Let $f \in \mathcal{R}$ and H be a homoclinic class of f whose index interval is $[i^-(H), i^+(H)]$. Properties (HC2) and (HC4) imply that for each $k \in [i^-(H), i^+(H)] \cap \mathbb{N}$ the set* $\mathrm{Per}(H)_k$ *is dense in the whole homoclinic class H.*

Using a notation similar to the one in Section 4.1.1, we denote by $\mathrm{Per}_{\mathbb{R}}(H)_k$ the subset of $\mathrm{Per}(H)_k$ consisting of saddles q such that all eigenvalues of $Df^{\tau(q)}(q)$ are real and different in modulus ($\tau(q)$ is the period of q).

(HC5) For every $f \in \mathcal{R}$, every non-trivial homoclinic class H of f, and every k in the index interval of H, the set $\mathrm{Per}_{\mathbb{R}}(H)_k$ is dense in H.

See [1, Proposition 2.3], which is just a dynamical reformulation of the results in [8, Lemma 4.16].

4.2.2. *Dominated splittings.* To prove Theorems 2 and 3, we consider the stable/unstable splittings naturally associated to saddles of a non-trivial homoclinic class. Given a diffeomorphism f in $\mathcal{R}$, a homoclinic class H of f, and k in the index interval of H, consider the Df-invariant *stable/unstable splitting over* $\mathrm{Per}_{\mathbb{R}}(H)_k$,

$$T_{\mathrm{Per}_{\mathbb{R}}(H)_k} M = E^s_k \oplus E^u_k,$$

where $E^s_k(p)$ and $E^u_k(p)$ are the stable and unstable bundles of $p \in \mathrm{Per}_{\mathbb{R}}(H)_k$. Note that $\dim(E^s_k(p)) = k$. In fact, the subscript k just emphasizes the s-index of the saddles we are considering.

If the stable/unstable splitting over $\mathrm{Per}_{\mathbb{R}}(H)_k$ is dominated, by item (1) in Proposition 2.1, it can be extended to the closure of $\mathrm{Per}_{\mathbb{R}}(H)_k$ in a dominated way. By property (HC5), the closure of $\mathrm{Per}_{\mathbb{R}}(H)_k$ is the whole H. Thus, in such a case, the homoclinic class H has a dominated splitting $E \oplus_< F$ with $\dim(E) = k$.

Finally, condition (HC2) and item (3) in Proposition 2.1 imply that for every g in $\mathcal{R}$ close to f, the continuation H_g of H for g also has a dominated splitting $E' \oplus_< F'$ with $\dim(E') = k$. Let us summarize these results:

Lemma 4.15. *Consider a diffeomorphism $f \in \mathcal{R}$, a homoclinic class H of f, and $k \in \mathbb{N}$ in the index interval of H. Suppose that the stable/unstable splitting over $\mathrm{Per}_{\mathbb{R}}(H)_k$ is dominated. Then for every $g \in \mathcal{R}$ the homoclinic class H_g has a dominated splitting $E' \oplus_< F'$ with $\dim(E') = k$.*

4.2.3. *Generation of homoclinic tangencies.* Given a diffeomorphism f and a (hyperbolic) saddle p of f, consider the set $\Sigma(p, f)$ of saddles q homoclinically related to p. The closure of $\Sigma(p, f)$ is the whole homoclinic class $H(p, f)$ of p. Note that if p has s-index k then $\Sigma(p, f) \subset \mathrm{Per}(H)_k$.

Lemma 4.16. *Consider a diffeomorphism $f \in \mathcal{R}$, a homoclinic class H of f, and k in the index interval of H.*

The stable/unstable splitting over $\mathrm{Per}_{\mathbb{R}}(H)_k$ is dominated if, and only if the stable/unstable splitting over $\Sigma(p, f)$ is dominated for every saddle $p \in \mathrm{Per}_{\mathbb{R}}(H)_k$.

Proof: By Lemma 4.15, if the stable/unstable splitting over $\mathrm{Per}_{\mathbb{R}}(H)_k$ is dominated then H has a dominated splitting $E \oplus_< F$ and $\dim(E) = k$, which is an extension of stable/unstable splitting over $\mathrm{Per}_{\mathbb{R}}(H)_k$. Thus the restriction of this dominated splitting over H to the subset $\Sigma(p, f)$

is also dominated. Moreover, since $\dim(E) = k$, it necessarily coincides with the stable/unstable splitting over $\Sigma(p, f) \subset \mathrm{Per}(H)_k$ (recall item (2) in Proposition 2.1). This concludes the proof of the first implication.

To prove the converse, take any saddle $p \in \mathrm{Per}_{\mathbb{R}}(H)_k$ and observe that, by (HC2), $H(p, f) = H$, thus $\Sigma(p, f)$ is dense in H. Hence, by item (1) in Proposition 2.1, the stable/unstable dominated splitting over $\Sigma(p, f)$ can be extended to a dominated splitting $E \oplus_< F$, $\dim(E) = k$, defined over the whole H. Thus $\mathrm{Per}_{\mathbb{R}}(H)_k \subset H$ also has a dominated splitting. Since $\dim(E) = k$ this splitting is necessarily the stable/unstable one over $\mathrm{Per}_{\mathbb{R}}(H)_k$. □

A key ingredient in the proof of Theorems 2 and 3 is the following result due to Gourmelon:

Proposition 4.17 (Theorem 6.1.1 in [25]). *Let p_f be a saddle of a diffeomorphism f such that the stable/unstable splitting defined over the set $\Sigma(p_f, f)$ of periodic points homoclinically related with p_f is not dominated. Then there is a diffeomorphism h arbitrarily C^1-close to f with a homoclinic tangency associated to p_h.*

A configuration that prevents the stable/unstable bundle over the set $\Sigma(p_f, f)$ to be dominated occurs when the angle between these bundles is not uniformly bounded from below. A specific configuration with this feature (to simplify the discussion we consider the three dimensional case) is the following: the homoclinic class of p_f contains saddles of s-indices one and two and with non-real multipliers. Under these hypotheses the stable/unstable bundle is not dominated. This sort of configuration can be found in [3].

Note that this result is a strong generalization of Proposition 4.2.

Proof of Theorem 3: Consider a diffeomorphism $f \in \mathcal{R}$, a non-trivial homoclinic class H of f, and $k \in \mathbb{N}$ in the index interval of H. By (HC5) the set $\mathrm{Per}_{\mathbb{R}}(H)_k$ is dense in $\mathrm{Per}(H)_k$. By item (1) in Proposition 2.1, and arguing as in Lemma 4.15, one has that the stable/unstable splitting over $\mathrm{Per}(H)_k$ is dominated if, and only if, the stable/unstable splitting over $\mathrm{Per}_{\mathbb{R}}(H)_k$ is dominated. Hence in the hypothesis of Theorem 3 we can replace $\mathrm{Per}(H)_k$ by its subset $\mathrm{Per}_{\mathbb{R}}(H)_k$.

Suppose that there is $k \in \mathbb{N}$ in the index interval of H such the stable/unstable dominated splitting over $\mathrm{Per}_{\mathbb{R}}(H)_k$ is not dominated. Take any saddle $p_f \in \mathrm{Per}_{\mathbb{R}}(H)_k$. By Remark 4.13, if g is close to f and belongs to $\mathcal{R}$ then $H_g = H(p_g, g)$ and $p_g \in \mathrm{Per}_{\mathbb{R}}(H_g)_k$.

By Lemma 4.16, the stable/unstable dominated splitting over $\Sigma(p_f, f)$ is not dominated. Therefore, by Proposition 4.17 there is a diffeomorphism h arbitrarily C^1 close to f with a homoclinic tangency associated to $p_h \in \mathrm{Per}_{\mathbb{R}}(H_h)_k$. Arguing as in Section 4.1.3, after an arbitrarily small perturbation, we can assume that this tangency occurs in a normally hyperbolic surface.

By Remark 4.10 to get diffeomorphisms g close to h (thus close to f) having horseshoes $\Lambda(p_g, n)$ with emerging entropy contained in the homoclinic class of p_g (note that these horseshoes are obtained using transverse intersections of the invariant manifolds of p_g). Therefore, for $g \in \mathcal{R}$ we have $\Lambda(p_g, n) \subset H_g = H(q_g, g)$. This means that we can argue exactly as in the proof of Theorem 1. This completes the sketch of the proof of Theorem 3. □

Proof of Theorem 2: It is enough to see that the hypotheses of the theorem implies that there is some k in the index interval $[i, j]$ of H such that stable/unstable dominated splitting over $\mathrm{Per}(H)_k$ is not dominated. Then Theorem 2 immediately follows from Theorem 3.

Assume that for every $r \in [i, j]$ the stable/unstable splitting $E^s_r \oplus E^u_r$ over $\mathrm{Per}(H)_r$ is dominated. By item (1) in Proposition 2.1, each dominated splitting $E^s_r \oplus_< E^u_r$ can be extended in a dominated way to the closure of $\mathrm{Per}(H)_r$,which is the whole H. Denote these extensions by $\bar{E}^s_r \oplus_< \bar{E}^u_r$.

Consider $i+1$ and the dominated splitting $\bar{E}^s_{i+1} \oplus_< \bar{E}^u_{i+1}$ of H with $\dim(\bar{E}^s_{i+1}) = i+1$. Using the dominated splitting $\bar{E}^s_i \oplus_< \bar{E}^u_i$ with $\dim(\bar{E}^s_i) = i$ and noting that $\bar{E}^s_i \subset \bar{E}^s_{i+1}$ we have

$$\bar{E}^s_{i+1} = \bar{E}^s_i \oplus_< E_1,$$

where $E_1 = \bar{E}^s_{i+1} \cap \bar{E}^u_i$ is one dimensional (for details see the arguments in [8, Lemma 4.12], uniqueness of the finest dominated splitting). In this way, we get a dominated splitting over H of the form

$$\bar{E}^s_i \oplus_< E_1 \oplus_< \bar{E}^u_{i+1},$$

with $\dim(E_1) = 1$.

The proof of the theorem now follows inductively. Arguing as above, we get that if the stable/unstable splitting over $\mathrm{Per}(H)_r$ is dominated for all $r = i, \ldots, i+k$, $i+k \leq j$, then the homoclinic class H has a dominated splitting of the form

$$\bar{E}^s_i \oplus_< E_1 \oplus_< \cdots \oplus_< \bar{E}^u_{r-1} \oplus_< \bar{E}^u_r.$$

Taking $r = j$, $\bar{E}^s_i = E^{cs}$, and $\bar{E}^u_j = E^{cu}$ it contradicts the hypotheses of Theorem 3: the homoclinic class has not an indecomposable central bundle of dimension at least 2 adapted to its s-indices. □

References

[1] F. Abdenur, C. Bonatti, S. Crovisier, L. J. Díaz, and G. Wen. Periodic points and homoclinic classes. *Ergod. Th. Dynamic. Systems*, 27(1):1–22, 2007.

[2] M. Asaoka. Hyperbolic sets exhibiting C^1-persistent homoclinic tangency for higher dimensions. *Proc. Amer. Math. Soc.*, 136(2):677–686, 2008.

[3] C. Bonatti and L. J. Díaz. Persistent nonhyperbolic transitive diffeomorphisms. *Ann. of Math. (2)*, 143(2):357–396, 1996.

[4] C. Bonatti and L. J. Díaz. Connexions hétéroclines et généricité d'une infinité de puits ou de sources. *Ann. Scient. Éc. Norm. Sup.*, 32:135–150, 1999.

[5] C. Bonatti and S. Crovisier. Recurrence et généricité. *Invent. Math.*, 158(1):33–104, 2004.

[6] C. Bonatti and L. J. Díaz. On maximal transitive sets of generic diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.*, 96:171–197, 2002.

[7] C. Bonatti, L. J. Díaz, and T. Fisher. Supergrowth of the number of periodic orbits for non-hyperbolic homoclinic classes. *Discrete and Cont. Dynamic. Systems*, 20(3):589–604, 2008.

[8] C. Bonatti, L. J. Díaz, and E. Pujals. A C^1-generic dichotomy for diffeomorphisms; weak forms of hyperbolicity or infinitely many sinks or sources. *Annals of Math.*, 158(355-418), 2003.

[9] C. Bonatti, L. J. Díaz, and M. Viana. *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective.*, volume 102 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005.

[10] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[11] Ch. Bonatti, L. J. Díaz, E.R. Pujals, and J. Rocha. Robustly transitive sets and heterodimensional cycles. *Astérisque*, 286:187–222, 2003.

[12] M. Boyle, D. Fiebig, and U. Fiebig. Residual entropy, conditional entropy, and subshift covers. *Forum Math.*, 14:713–757, 2002.

[13] M. Brin and Stuck G. *Introduction to Dynamical Systems*. Cambridge University Press, 2002.

[14] D. Burguet. C^2 surface diffeomorphisms have symbolic extensions. preprint.

[15] K. Burns. personal communication.

[16] J. Buzzi. Intrinsic ergodicity for smooth interval maps. *Isreal J. Math.*, 100:125–161, 1997.

[17] J. Buzzi, T. Fisher, M. Sambarino, and C. Vásquez. Intrinsic ergodicity for certain partially hyperbolic derived from Anosov systems. preprint.

[18] A. Candel and L. Conlon. *Foliations I*, volume 23 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000.

[19] C. Carbalo, C. Morales, and M. J. Pacifico. Homoclinic classes for generic C^1 vector fields. *Ergod. Th. Dynamic. Systems*, 23(2):403–415, 2003.

[20] W. Cowieson and L.-S. Young. SRB measures as zero-noise limits. *Ergod. Th. Dynamic. Systems*, 25(4):1115–1138, 2005.

[21] L. J. Díaz, E. Pujals, and R. Ures. Partial hyperbolicity and robust transitivity. *Acta Math.*, 183:1–43, 1999.

[22] T. Downarowicz and A. Maass. Smooth interval maps have symbolic extensions. *Inventiones Math.*, 176(3):617–636, 2009.

[23] T. Downarowicz and S. Newhouse. Symbolic extensions in smooth dynamical systems. *Inventiones Math.*, 160(3):453–499, 2005.

[24] N. Gourmelon. Difféomorphismes robustement transitifs. Master's thesis, Univ. Grenoble, 2003.
[25] N. Gourmelon. *Instabilité de la dynamique en lâbsence de décomposition dominée*. PhD thesis, Univ. de Bourgogne, 2006.
[26] S. Hayashi. Connecting invariant manifolds and the solution of the C^1 stability and ω-stability conjectures for flows. *Annals of Math.*, 145:81–137, 1997.
[27] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.
[28] G. Keller. *Equilibrium states in ergodic theory*. London Mathematical Society Student Texts. Cambridge University Press, 1998.
[29] R. Mañé. Contributions to the stability conjecture. *Topology*, 17:383–396, 1978.
[30] M. Misiurewicz. Diffeomorphim without any measure of maximal entropy. *Bull. Acad. Pol. Sci., ser sci. math, astr. et phys*, 21:903–910, 1973.
[31] S. Newhouse. Hyperbolic limit sets. *Trans. Amer. Math. Soc.*, 167:125–150, 1972.
[32] S. Newhouse. Continuity properties of entropy. *Annals of Math.*, 129:215–235, 1989.
[33] M. J. Pacifico and J. Vieitez. Robust entropy-expansiveness implies generic domination. Preprint.
[34] M. J. Pacifico and J. Vieitez. Entropy-expansiveness and domination for surface diffeomorphisms. *Rev. Mat. Complut.*, 21(2):293–317, 2008.
[35] E. Pujals and M. Sambarino. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. *Annals of Math.*, 151:961–1023, 2000.
[36] R. Saghin and Z. Xia. The entropy conjecture for partially hyperbolic diffeomorphisms with 1-D center. *Topology Appl.*, 157(1):29–34, 2010.
[37] M. Shub. Dynamical systems, filtrations and entropy. *Bull. Amer. Math. Soc.*, 80:27–41, 1974.
[38] M. Shub. *Global Stability of Dynamical Systems*. Springer- Verlag, New York, 1987.
[39] K. Sigmund. Generic properties of invariant measures for Axiom-A-diffeomorphisms. *Inventiones Math.*, 11:99–109, 1970.
[40] C. P. Simon. Instability in Diff(T^3) and the nongenericity of rational zeta function. *Trans. Amer. Math. Soc.*, 174:217–242, 1972.
[41] P. Walters. *An Introduction to Ergodic Theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin-New York, 1982.

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