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SUPER-EXPONENTIAL GROWTH OF THE NUMBER OF PERIODIC ORBITS INSIDE HOMOCLINIC CLASSES

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Abstract. We show there is a residual subset $\mathcal{S}(M)$ of $\mathrm{Diff}^1(\mathrm{M})$ such that, for every $f \in \mathcal{S}(M)$, any homoclinic class of f containing periodic saddles p and q of indices α and β, respectively, where $\alpha < \beta$, has superexponential growth of the number of periodic points inside the homoclinic class. Furthermore, it is shown the super-exponential growth occurs for hyperbolic periodic points of index γ inside the homoclinic class for every $\gamma \in [\alpha, \beta]$.

1. **Introduction.**

A diffeomorphism f is Artin-Mazur (A-M for short) if the number of isolated periodic points of period n of f, denoted by $\mathbb{P}_n(f)$, grows at most exponentially fast: there is a constant $C > 0$ such that

$$\# \mathbb{P}_k(f) \leq \exp(C\, k), \quad \text{for all } k \in \mathbb{N}.$$

Artin and Mazur proved in [5] that the A-M maps are dense in the space of C^r-maps endowed with the uniform topology. Later, Kaloshin proved in [25] that the A-M diffeomorphisms having only hyperbolic periodic points are dense in the space of C^r-diffeomorphisms.

Later, in [26], Kaloshin proved that the A-M diffeomorphisms are not topologically generic in the space of C^r-diffeomorphisms ($r \geq 2$). The proof of this result involves the so-called Newhouse domains (i.e., an open set where the diffeomorphisms with homoclinic tangencies are dense). The standard way to get Newhouse domains is by unfolding a homoclinic tangency of a C^2-diffeomorphism, see [35]. For three-dimensional volume-preserving maps the existence of Newhouse domains and the genericity of super-exponential growth for C^r maps, where $2 \leq r < \infty$, was established by Kaloshin and Saprykina [31]. Additionally, Kaloshin and Kozlovsky [30] were able to construct a surprising example of a one-dimensional unimodal C^r map, for $2 \leq r < \infty$, with super-exponential growth of the periodic points.

The main technical step in [26] is the following. An open set $\mathcal{K}$ has a *super-exponential growth for the number of periodic points* if for every arbitrary sequence

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of positive integers $a = (a_k)_{k=1}^{\infty}$ there is a residual subset $\mathcal{R}(a)$ of $\mathcal{K}$ such that $\limsup_{k\to\infty} \#\mathbb{P}_k(f)/a_k = \infty$ for any diffeomorphism $f \in \mathcal{R}(a)$. In [26] it was proved that Newhouse domains have super-exponential growth for the number of periodic points. Moreover, given any Newhouse domain $\mathcal{N}$, there is a dense subset $\mathcal{D}$ of $\mathcal{N}$ of diffeomorphisms having a curve of periodic points.

To get a dynamical configuration, in the C^1-topology, generating (with some persistence) curves of periodic points is quite simple. One can proceed as follows, consider a diffeomorphism f defined on a manifold M of dimension n, $n \geq 3$, having a non-hyperbolic homoclinic class $H(p_f, f)$ such that for every g in a C^1-neighborhood $\mathcal{U}_f$ of f the homoclinic class $H(p_g, g)$ (p_g is the continuation of p_f for g) contains a saddle q_g whose *index* (dimension of the unstable bundle) is different from the one of p_f. Then there is a dense subset $\mathcal{D}$ of $\mathcal{U}_f$ consisting of diffeomorphisms g having a saddle-node. Moreover, the period of such a saddle-node can be taken arbitrarily big. Using this fact, and noting that the behavior of a saddle-node in the non-hyperbolic direction is close to the identity, one gets after a new perturbation, a diffeomorphism having an interval of fixed periodic points. The proof of these properties follows using the arguments in [2], which are refinements of the constructions in [12] (we will review this construction in Section 4). Examples of homoclinic classes containing with some persistence saddles with different indices can be found in [9, 17, 18, 19].

A more interesting fact is that in some cases the periodic points generating the super-growth of the number of periodic points can be obtained *inside* the homoclinic class, thus generating homoclinic classes that have super-exponential growth for the number of periodic points.

Before stating our main result let us recall that *homoclinic classes* were introduced by Newhouse in [34] as a generalization of the basic sets in the Smale Decomposition Theorem (see [36]). The homoclinic class of a hyperbolic saddle p of a diffeomorphism f, denoted by $H(p, f)$, is the closure of the transverse intersections of the invariant manifolds (stable and unstable ones) of the orbit of p. A homoclinic class can be also (equivalently) defined as the closure of the set of hyperbolic saddles q *homoclinically related* to p (the stable manifold of the orbit of q transversely meets the unstable one of the orbit of p and vice-versa). We assume throughout that M is a compact boundaryless smooth manifold.

Let

$$\mathbb{P}_k^{\gamma}(H(p, f)) = \{x \in H(p, f),\ f^k(x) = x,\ x \text{ hyperbolic and index } (x) = \gamma\}.$$

Motivated by the definitions above, we say that the *saddles of index γ of the homoclinic class $H(p_f, f)$ have a super-exponential growth in $\mathcal{U}$* if for every sequence of positive integers $a = (a_k)_{k=1}^{\infty}$ there is a residual subset $\mathcal{R}(a)$ of $\mathcal{U}$ such that the *growth of the number of saddles of index γ of the homoclinic class is lower bounded by the sequence* $a = (a_k)$, that is,

$$\limsup_{k\to\infty} \frac{\#\,\mathbb{P}_k^{\gamma}(H(p, f))}{a_k} = \infty, \quad \text{for every diffeomorphism } f \in \mathcal{R}(a).$$

Theorem 1. *There is a residual subset $\mathcal{S}(M)$ of* $\mathrm{Diff}^1(\mathrm{M})$ *of diffeomorphisms f such that, for every $f \in \mathcal{S}(M)$, any homoclinic class of f containing hyperbolic saddles of different indices has super-exponential growth of the number of periodic points.*

In fact, a stronger version of this theorem holds:

Theorem 2. *The residual subset $\mathcal{S}(M)$ of* $\mathrm{Diff}^1(\mathrm{M})$ *in Theorem 1 can be chosen as follows: Consider $f \in \mathcal{S}(M)$ and any homoclinic class $H(p,f)$ of f containing saddles of different indices α and β, $\alpha < \beta$. Then for every natural number $\gamma \in [\alpha, \beta]$ the number of (hyperbolic) periodic points of index γ of $H(p,f)$ has super-exponential growth.*

The novelty of these results is that the super-exponential growth of periodic orbits occurs inside the non-hyperbolic homoclinic classes. In fact, our results are localization of the following result. Recall that a diffeomorphism f is called a *star diffeomorphism*, denoted $f \in \mathcal{F}^*(M)$, if it has a neighborhood $\mathcal{U}$ in $\mathrm{Diff}^1(M)$ such that every periodic of every diffeomorphism $g \in \mathcal{U}$ is hyperbolic (this is the so-called *star condition*). For C^1-diffeomorphisms, the star condition is equivalent to the Axiom A plus no-cycles condition, see [32, 33] for surface diffeomorphisms and [3, 23] for higher dimensions. It is not hard to prove that C^1-generic diffeomorphisms in the complement of the star diffeomorphisms have super-exponential growth of periodic points: it is enough to get periodic points having an eigenvalue equal to one, perturbing such a non hyperbolic periodic point one gets an arbitrarily large number of periodic orbits having the same period as the non-hyperbolic one. The second step is to check that the period of these non-hyperbolic orbits can be taken arbitrarily large. Note that this argument does not localize the super-growth of periodic orbits inside a given class. This is exactly the novelty of our constructions.

Finally, these two results are consequences of the following result. Consider an open set $\mathcal{U}$ of $\mathrm{Diff}^1(\mathrm{M})$ such that for every f in $\mathcal{U}$ there are hyperbolic periodic saddles p_f and q_f, depending continuously on f, having different indices. By [2, Lemma 2.1] (see also property (G3) in Section 2) there is a residual subset $\mathcal{G}$ of $\mathcal{U}$ such that

- either $H(p_f, f) = H(q_f, f)$ for all $f \in \mathcal{G}$,
- or $H(p_f, f) \cap H(q_f, f) = \emptyset$ for all $f \in \mathcal{G}$.

In the first case, we say that the saddles p_f and q_f are *generically homoclinically linked* in $\mathcal{U}$.

Note that this relation is different from the usual homoclinic relation: being homoclinically related is an open property and saddles homoclinically related have the same index, while saddles being generically homoclinically linked may have different indices.

The next proposition is then sufficient to prove Theorem 2.

Proposition 1. *Consider an open set $\mathcal{U}$ of* $\mathrm{Diff}^1(\mathrm{M})$ *and a pair of saddles p_f and q_f which are generically homoclinically linked in $\mathcal{U}$. Suppose that the indices of the saddles p_f and q_f are α and β, $\alpha < \beta$. Then, for every natural number $\gamma \in [\alpha, \beta]$, the saddles of index γ of $H(p_f, f)$ have super-exponential growth in $\mathcal{U}$.*

We now need that notion of a *chain recurrence class.* A point y is *f-chain attainable* from the point x if for every $\varepsilon > 0$ there is an *ε-pseudo-orbit* going from x to y. Recall that an ε-pseudo-orbit of a diffeomorphism f is a family $(x_i)_{i=0}^n$ of points such that $\mathrm{dist}\,(f(x_i), x_{i+1}) < \varepsilon$ for all $i = 0, \ldots, n-1$. The points x and y are *f-bi-chain attainable* if x is chain attainable from y and vice-versa. The bi-chain attainability relation defines an equivalence relation on the *chain recurrent set* $R(f)$ of f (i.e., the set of points x which are chain attainable from themselves). The *chain recurrence classes* are the equivalence classes of $R(f)$ for the bi-chain attainability relation.

PSfrag replacements

r_f

q_f

s_f

Figure 1. Heteroclinic intersections

A diffeomorphism f is *tame* if every chain recurrence class of f is robustly isolated. In this case, for diffeomorphisms in a residual subset of $\mathrm{Diff}^1(\mathrm{M})$, these chain recurrence classes are in fact homoclinic classes, see [8]. In fact, one also has that non-hyperbolic homoclinic classes of C^1-generic tame diffeomorphisms contain saddles of different indices, see [2]. Thus Proposition 1 implies the following:

Corollary 1. *Every non-hyperbolic homoclinic class of a C^1-generic tame diffeomorphism has super-exponential growth of the number of periodic points.*

This paper can be viewed as a continuation of [2], where it is proved that, for C^1-generic diffeomorphisms, the indices of the saddles in a homoclinic class form an interval in $\mathbb{N}$: there is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(\mathrm{M})$ such that for every $f \in \mathcal{R}$ and every homoclinic class $H(p, f)$ of f, if the class contains saddles of indices α and β, $\alpha < \beta$, it is also contains saddles of indices γ for all $\gamma \in [\alpha, \beta] \cap \mathbb{N}$. In fact, our proof follows from the constructions in [2]. We proceed as follows, suppose that the homoclinic class $H(p_f, f)$ contains saddles q_f of index α and r_f of index $\alpha + 1$. By the results in [8, 16] (homoclinic classes of C^1-generic diffeomorphisms either coincide or they are disjoint, see also property (G2) in Section 2), we can assume that $H(p_f, f) = H(q_f, f) = H(r_f, f)$. We see that, after a perturbation, there is a saddle-node s_f (of arbitrarily large period) with $s_f \in H(q_f, f) = H(r_f, f)$. This saddle-node has a strong stable direction of the same dimension as the stable one of r_f (i.e., of dimension $n - \alpha - 1$, n is the dimension of the ambient manifold), a strong unstable direction of the same dimension as the unstable one of q_f (i.e., dimension α), and a one dimensional central direction. We see that the strong stable manifold of s_f transversely meets the unstable manifold of r_f and that the strong unstable manifold of s_f transversely meets the stable manifold of q_f, see Figure 1. This will imply that the these three saddles belong to the same chain recurrence class of f. Unfolding the saddle-node, we get a diffeomorphism g with new (hyperbolic) saddles of indices α and $\alpha + 1$ which are in the same chain recurrence class (the one of r_g and q_g). In fact, we can get an arbitrarily large number of such saddles. Let us explain this point.

Consider now a sequence $a = (a_k)_{k=1}^{\infty}$ of strictly positive integers. The previous constructions, specially the fact that the period of s_f can be taken arbitrarily large, imply that, after a small perturbation, we can assume that the dynamics of f at the saddle-node s_f in the central direction is exactly the identity. Suppose that the period of the saddle-node is k (large k). We fix a_k and perturb the diffeomorphism f to get a new diffeomorphism g such that g has $k\, a_k$ saddles $q_g^1, \dots, q_g^{k\, a_k}$ of index α and a_k saddles $r_g^1, \dots, r_g^{k\, a_k}$ of index $\alpha + 1$ of the same period as s_f and arbitrarily close to s_f. By a continuity argument, this implies that, for every i, the unstable manifolds of q_g^i and r_g^i meets the stable manifold of q_g and the stable manifolds of q_g^i and r_g^i meets the unstable manifold of q_g. Next step is to check the saddles q_g^i and r_g^i are in the chain recurrence class of r_g and q_g. The argument now follows noting that for C^1-generic diffeomorphisms every chain recurrence class containing a saddle is the homoclinic class of such a saddle, see [8, Remarque 1.10] and (G2) in Section 2.

The previous arguments show that for each $k \in \mathbb{N}$ there exists a residual set $\mathcal{G}(k)$ such that for each $g \in \mathcal{G}(k)$ there exists $n_g(k) \geq k$ where $H(p_g, g)$ has $n_g(k) a_{n_g(k)}$ different periodic points of index γ and $n_g(k)\, a_{n_g(k)}$ different periodic points of index $\gamma + 1$. Consider now the set $\mathcal{R}(a)$ defined as the intersection of the sets $\mathcal{G}(k)$. By construction, the set $\mathcal{R}(a)$ is residual in $\mathcal{U}$ and consists of diffeomorphisms such that the growth of saddles of indices α and $\alpha + 1$ is lower bounded by the sequence $a = (a_k)_{k=1}^{\infty}$. This completes the sketch of the proof.

1.1. Open Questions.

One question concerns a dichotomy for C^1-generic dynamical systems of exponential versus super-exponential growth of periodic points.

Residually among C^1-diffeomorphisms the homoclinic classes depend continuously on the diffeomorphism and two homoclinic classes are either disjoint or coincide, see [8, 16]. Let $\mathcal{R}$ be such a residual set. For $f \in \mathcal{R}$ let $\#(f) \in \mathbb{N} \cup \infty$ be the number of homoclinic classes of f. The function $\#(f)$ is lower semi-continuous and in fact $\mathcal{R}$ can be chosen such that the map $\#(\cdot)$ is well defined and locally constant, see [1]. A diffeomorphism $f \in \mathcal{R}$ is *wild* if there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{R}$ such that $\#(g) = \infty$ for all $g \in \mathcal{U}$. By [10], the set of wild diffeomorphisms is not empty in $\mathrm{Diff}^1(M)$ when the dimension of M is equal to or greater than three. It can be shown that for tame diffeomorphisms in $\mathcal{R}$ there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{R}$ such that $\#(g) < \infty$ for all $g \in \mathcal{U}$.

From Corollary 1, we know that for C^1-generic tame diffeomorphisms every homoclinic class is either hyperbolic or has a super-exponential growth of the number of periodic points. The following conjecture can then be shown by proving the same property for wild diffeomorphisms.

Conjecture 1. *There is a C^1-generic dichotomy for diffeomorphisms: either the homoclinic classes are hyperbolic or there is super-exponential growth of the number of periodic points.*

Another question concerns the existence of symbolic extensions of diffeomorphisms. A *symbolic extension* of a $f \in \mathrm{Diff}^1(\mathrm{M})$ consists of a shift space (Σ, σ) (not necessarily of finite type, but with a finite alphabet) and a continuous surjection $\pi : \Sigma \to M$ (not necessarily finite-to-one) such that the following diagram commutes.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \pi\downarrow & & \downarrow\pi \\ M & \xrightarrow[f]{} & M. \end{array}$$

The study of symbolic extensions for a given system leads to the Downarowicz theory of an *entropy structure,* see [20]. An *entropy structure* is a master entropy invariant that provides a precise picture of entropy arising on finer and finer scales.

Unfortunately, at this point it is unclear which diffeomorphisms have a symbolic extension. One class of maps that are known to have symbolic extensions are C^∞

diffeomorphisms of a compact smooth manifold into itself. This follows from results of Buzzi in [15], Boyle, D. Fiebig, and U. Fiebig in [14], and Boyle and Downarowicz in [13]. In this case, the system has a symbolic extension given by a factor map which preserves entropy for every invariant measure.

However, in [21] it is shown that among area preserving non-Anosov diffeomorphisms of a smooth compact surface there exists a C^1-residual set $\mathcal{S}$ such that each $f \in \mathcal{S}$ has no symbolic extension. The idea is that on "finer and finer" scales one sees "more" entropy appearing. The construction in this paper shows on finer scales one sees a super-exponential growth of periodic orbits, but we do not necessarily see a growth in entropy due to the construction. Adapting the construction of this paper it may be possible to answer the following question in the affirmative.

Question 1. Among diffeomorphisms containing a homoclinic class with periodic points of differing index is there a C^1-residual set $\mathcal{S}$ such that any $f \in \mathcal{S}$ does not have a symbolic extension?

In a recent work [7] by Asaoka, it is shown that on any manifold of dimension greater than or equal to three there is a C^1-open set of diffeomorphisms containing a generic set $\mathcal{R}$ such that each $f \in \mathcal{R}$ does not have a symbolic extension. Asaoka shows there is a C^1-generic set with infinitely many sinks and uses techniques in [21] to show there are no symbolic extensions. However, the construction in [7] only works for a very specific sort of homoclinic classes. We then have the following question.

Question 2. If $\mathcal{U}$ is a C^1-open set containing a residual subset $\mathcal{R}$ of diffeomorphisms containing an infinite number of sinks, is there a residual set $\mathcal{R}' \subset \mathcal{R}$ such that each $f \in \mathcal{R}'$ does not have a symbolic extension?

A further question is how typical superexponential growth is in a measure theoretic sense. There are well known examples of sets in Euclidean space that are generic, but have Lebesgue measure zero. The notion of prevalence is a measure theoretic notion of genericity. (See [24] for a definition of prevalence.) In a series of papers [27, 28, 29] Kaloshin and Hunt have investigated the growth of periodic points for prevalent diffeomorphisms. They have shown that the growth is not much faster than exponential. Specifically, for each $\epsilon, \delta > 0$ there exists a prevalent set of $C^{1+\epsilon}$ diffeomorphisms (or smoother) for which the number of periodic points is bounded above by $e^{Cn^{1+\delta}}$. Along these lines Arnold has posed the following problem [4]:

Problem. *Prove that for a generic finite parameter family of diffeomorphisms $\{f_\epsilon\}$, for Lebesgue almost every ϵ we have that f_ϵ is A-M.*

This paper is organized as follows. In Section 2, we review the properties of C^1-generic diffeomorphisms concerning homoclinic classes. We also state some perturbations lemmas we use throughout the paper. In Section 3, we prove Proposition 1. The main step of this proof is Proposition 2 which analyzes the growth of the number of periodic points inside homoclinic classes of generic diffeomorphisms having saddles of different indices. To prove Proposition 2, in Section 4 we study the generation of saddles via heterodimensional cycles.

2. **Generic properties of C^1-diffeomorphisms and perturbation lemmas.** Using recent results on the dynamics of C^1-generic diffeomorphisms, one gets a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(\mathrm{M})$ of diffeomorphisms f verifying properties (G1)–(G5) below:

(G1): The periodic points of f are dense in the chain recurrent set of f. In particular, the non-wandering set and the chain recurrent set of f coincide, see [8, Corollaire 1.2]

(G2): Every chain recurrence class Λ of f containing a (hyperbolic) periodic point p satisfies $\Lambda = H(p, f)$. In particular, for any pair of saddles p and q of f, either $H(p, f) = H(q, f)$ or $H(p, f) \cap H(q, f) = \emptyset$, see [8, Remarque 1.10] (for previous results see [6, 16]).

In what follows, if p_f is a hyperbolic periodic point of a diffeomorphism f, we denote by p_g the continuation of p_f for g close to f. We have the following refinement of (G2):

(G3): For every pair of saddles p_f and q_f of f, there is a neighborhood $\mathcal{U}_f$ of f in $\mathcal{G}$ such that either $H(p_g, g) = H(q_g, g)$ for all $g \in \mathcal{U}_f$, or $H(p_g, g) \cap H(q_g, g) = \emptyset$ for all $g \in \mathcal{U}_f$, see [2, Lemma 2.1].

(G4): For every saddle p_f of f whose homoclinic class $H(p_f, f)$ contains saddles of indices s and $s + k$, there is a neighborhood $\mathcal{U}_f$ of f in $\mathcal{G}$ such that, for every $g \in \mathcal{U}_f$, the homoclinic class $H(p_g, g)$ contains saddles $q_g^s, q_g^{s+1}, \cdots, q_g^{s+k}$ of indices $s, s + 1$, and $s + k$, see [2, Theorem 1].

Given a homoclinic class $H(p, f)$, denote by $\mathrm{Per}_h(H(p, f))$ the set of hyperbolic saddles q *homoclinically related* to p, that is, the stable manifold of the orbit of q has a point of transverse intersection with the unstable manifold of the orbit of p and vice-versa. Note that homoclinically related saddles have the same index. The set $\mathrm{Per}_h(H(p, f))$ is dense in $H(p, f)$, see [34].

We say that a periodic point p of period $\pi(p)$ of a diffeomorphism f has *real multipliers* if every eigenvalue of the linear isomorphism $Df^{\pi(p)}(p) \colon T_pM \to T_pM$ is real, positive, and has multiplicity one. We denote by $\mathrm{Per}_{\mathbb{R}}(H(p, f))$ the subset of $\mathrm{Per}_h(H(p, f))$ of periodic points with real multipliers.

(G5): For every diffeomorphisms $f \in \mathcal{G}$ and every nontrivial homoclinic class $H(p, f)$ of f, the set $\mathrm{Per}_{\mathbb{R}}(H(p, f))$ is dense in $H(p, f)$, see [2, Proposition 2.3], which is just a dynamical reformulation of [11, Lemmas 1.9 and 4.16].

We close this section quoting some standard perturbations lemmas in C^1-dynamics. The first one allows us to perform dynamically the perturbations of cocycles:

Lemma 1. (Franks, [22]**).** *Consider a diffeomorphism f and an f-invariant finite set Σ. Let A be an ε-perturbation of the derivative of f in Σ (i.e., the linear maps $Df(x)$ and $A(x)$ are ε-close for all $x \in \Sigma$). Then, for every neighborhood U of Σ, there is a diffeomorphism g ε-C^1-close to f such that*

- *$f(x) = g(x)$ for every $x \in \Sigma$ and every $x \notin U$,*
- *$Dg(x) = A(x)$ for all $x \in \Sigma$.*

The next lemma allows us to obtain intersections between invariant manifolds of periodic points.

Lemma 2. (Hayashi's Connecting Lemma, [23]**)** *Let p_f and q_f be a pair of saddles of a diffeomorphism f such that there are sequences of points y_n and of natural numbers k_n such that*

- $y_n \to y \in W^u_{loc}(p_f, f)$, $y \neq p_f$, *and*
- $f^{k_n}(y_n) \to z \in W^s_{loc}(q_f, f)$, $z \neq p_f$.

Then there is a diffeomorphism g arbitrarily C^1-close to f such that $W^u(p_g, g)$ and $W^s(q_g, g)$ have an intersection arbitrarily close to y.

Recall that a diffeomorphism f has a *heterodimensional cycle* associated to the saddles p and q if p and q have different indices and both intersections $W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$ and $W^s(\mathcal{O}_q) \cap W^u(\mathcal{O}_p)$ are non-empty. An immediate consequence of the connecting lemma is the following:

Lemma 3. *Let $\mathcal{U}$ be an open set of* $\mathrm{Diff}^1(\mathrm{M})$ *such that the saddles p_f and q_f of indices α and β, $\alpha < \beta$, are generically homoclinically linked in $\mathcal{U}$. There there is a dense subset $\mathcal{D}$ of $\mathcal{U}$ of diffeomorphisms f having a heterodimensional cycle associated to p_f and q_f.*

Proof: This result follows from applying the Connecting Lemma twice to the diffeomorphisms in $\mathcal{U}$. First, using the transitivity of the homoclinic class $H(p_f, f)$, one gets a dense subset $\mathcal{T}$ of $\mathcal{U}$ such that the unstable manifold of the orbit of q_f and the stable manifold of the orbit of p_f have non-empty intersection. As the sum of the dimensions of these manifolds is strictly greater than the dimension of the ambient, one can assume that this intersection is transverse. Thus such an intersection persists by perturbations. Hence, the set $\mathcal{T}$ contains an open and dense subset $\mathcal{S}$ of $\mathcal{U}$ such that $W^s(\mathcal{O}_{p_f})$ transversally meets $W^u(\mathcal{O}_{q_f})$, for every $f \in \mathcal{S}$. A new application of the Connecting Lemma, now interchanging the roles of p_f and q_f, gives a dense subset $\mathcal{D}$ of $\mathcal{S}$ (thus a dense subset of $\mathcal{U}$) such that $W^s(\mathcal{O}_{q_f}) \cap W^u(\mathcal{O}_{p_f}) \neq \emptyset$. Thus every diffeomorphisms $f \in \mathcal{D}$ has a heterodimensional cycle associated to p_f and q_f. For details see [2, Section 2.4]. $\square$

3. **Super-growth of periodic orbits in non-hyperbolic homoclinic classes.**

In this section, we prove Proposition 1. The main step of this proof is the following:

Proposition 2. *Let $a = (a_k)_{k=1}^{\infty}$ be a sequence of natural numbers and $\mathcal{U}$ an open set of* $\mathrm{Diff}^1(\mathrm{M})$ *such that there are saddles p_f and q_f of different indices which are generically homoclinically linked in $\mathcal{U}$.*

Let α and β, $\alpha < \beta$, be the indices of the saddles p_f and q_f. Then, for every $\gamma \in [\alpha, \beta] \cap \mathbb{N}$ and every $k \in \mathbb{N}$, there is a residual subset $\mathcal{G}^\gamma(k)$ of $\mathcal{U}$ such that for every diffeomorphism $\varphi \in \mathcal{G}^\gamma(k)$ there is $n_\varphi(k) \geq k$ such that the homoclinic class $H(p_\varphi, \varphi)$ contains at least $n_\varphi(k)\, a_{n_\varphi(k)}$ different periodic orbits of period $n_\varphi(k)$ and index γ.

Proof of Proposition 1: Proposition 1 is a straightforward consequence of Proposition 2. Consider any sequence $a = (a_k)_{k=1}^{\infty}$ of natural numbers and fix a natural number $\gamma \in [\alpha, \beta]$. Consider the intersection

$$\mathcal{R}^\gamma = \mathcal{R}^\gamma(a) = \bigcap_k \mathcal{G}^\gamma(k).$$

By construction, this set is residual in $\mathcal{U}$. We claim that, for every $\varphi \in \mathcal{R}^\gamma$, it holds

$$\limsup_{k\to\infty} \frac{\#\, \mathbb{P}^\gamma_k(H(p_\varphi, \varphi))}{a_k} = \infty.$$

Since $\varphi \in \mathcal{R}^\gamma$, one has that $\varphi \in \mathcal{G}^\gamma(k)$ for all $k \in \mathbb{N}$. Thus, for each k, there is $n_k(\varphi) \geq k$ such that the homoclinic class of $H(p_\varphi, \varphi)$ contains at least $n_\varphi(k)\, a_{n_\varphi(k)}$

periodic orbits of index γ and period $n_\varphi(k)$. As $n_k(\varphi) \to \infty$, there is a strictly increasing subsequence (n_{k_j}) of $(n_k(\varphi))$ with $n_{k_j} \to \infty$ and such that

$$\frac{\# \mathbb{P}^{\gamma}_{n_{k_j}}(H(p_\varphi, \varphi))}{a_{n_{k_j}}} \geq n_{k_j}.$$

This implies our claim.

Taking the residual subset $\mathcal{R}(a)$ of $\mathcal{U}$ defined by

$$\mathcal{R}(a) = \bigcap_{\gamma=\alpha}^{\beta} \mathcal{R}^{\gamma}(a)$$

one concludes the proof of the proposition. □

Proof of Proposition 2: Let $\mathcal{G}$ be the residual subset of $\mathrm{Diff}^1(\mathrm{M})$ in Section 2 and write $\mathcal{G}_{\mathcal{U}} = \mathcal{G} \cap \mathcal{U}$ (this set is residual in $\mathcal{U}$).

Lemma 4. *Let $\mathcal{U}$ be an open set such that there are saddles p_f and q_f of indices α and β, $\alpha < \beta$, which are generically homoclinically linked in $\mathcal{U}$. Then for every $g \in \mathcal{G}_{\mathcal{U}}$ there is a neighborhood $\mathcal{V}_g$ in $\mathcal{G}_{\mathcal{U}}$ such that for every $\varphi \in \mathcal{V}_g$ there are saddles $q^{\alpha}_{\varphi}, q^{\alpha+1}_{\varphi}, \ldots, q^{\beta}_{\varphi}$ such that:*

- *$H(p_g, g) = H(q^{\alpha}_g, g) = \cdots = H(q^{\beta}_g, g) = H(q_g, g)$,*
- *the index of q^i_φ is i,*
- *every saddle q^i_φ has real multipliers, and*
- *the saddles q^i_φ depend continuously on φ.*

Proof: By generic properties (G2), (G3), and (G4), for every $\varphi \in \mathcal{G}_{\mathcal{U}}$ there are saddles $q^{\alpha}_{\varphi}, q^{\alpha+1}_{\varphi}, \ldots, q^{\beta}_{\varphi}$ of indices $\alpha, \alpha+1, \ldots, \beta$ such that

$$H(p_\varphi, \varphi) = H(q^{\alpha}_\varphi, \varphi) = \cdots = H(q^{\beta}_\varphi, \varphi) = H(q_\varphi, \varphi).$$

Moreover, by (G3), these saddles can be chosen depending continuously on a small neighborhood of φ. Finally, by (G5), we can assume that these saddles have real multipliers. This concludes the proof of the lemma. □

Given a pair of hyperbolic periodic points q and p, we write $q <_{\mathrm{us}} p$ if the unstable manifold $W^u(\mathcal{O}_q)$ of the orbit $\mathcal{O}_q$ of q intersects transversally the stable manifold $W^s(\mathcal{O}_p)$ of the orbit $\mathcal{O}_p$ of p: there exists a point $x \in W^u(\mathcal{O}_q) \cap W^s(\mathcal{O}_p)$ such that $T_xM = T_xW^u(\mathcal{O}_q) + T_yW^s(\mathcal{O}_p)$.

Remark 1. The property $<_{\mathrm{us}}$ is open in $\mathrm{Diff}^1(\mathrm{M})$: let p_f and q_f be hyperbolic periodic points of a diffeomorphism f with $q_f <_{\mathrm{us}} p_f$, then there is a neighborhood $\mathcal{V}_f$ of f in $\mathrm{Diff}^1(\mathrm{M})$ such that $q_g <_{\mathrm{us}} p_g$ for every $g \in \mathcal{V}_f$.

The main step of the proof of Proposition 2 is the following:

Proposition 3. *Let $a = (a_k)_{k=1}^{\infty}$ be a sequence of natural numbers. Let f be a diffeomorphism having a pair of hyperbolic periodic saddles p_f and q_f with real multipliers. Assume that the indices of p_f and q_f are γ and $\gamma+1$ and that f has a heterodimensional cycle associated to p_f and q_f.*

Then for every $k \in \mathbb{N}$ there are $n_k \geq k$ and a diffeomorphism g arbitrarily C^1-close to f having $n_k\, a_{n_k}$ saddles $r^{\gamma}_1, \ldots, r^{\gamma}_{n_k\, a_{n_k}}$ of period n_k and index γ and $n_k\, a_{n_k}$ saddles $r^{\gamma+1}_1, \ldots, r^{\gamma+1}_{n_k\, a_{n_k}}$ of period n_k and index $\gamma+1$ such that

$$q_g <_{\mathrm{us}} r^{\gamma}_i <_{\mathrm{us}} p_g \quad \text{and} \quad q_g <_{\mathrm{us}} r^{\gamma+1}_i <_{\mathrm{us}} p_g, \quad \text{for all } i = 1, \ldots, n_k\, a_{n_k}.$$

Moreover, the orbits of the saddles $r_1^\gamma, \dots, r^\gamma_{n_k\, a_{n_k}}, r_1^{\gamma+1}, \dots, r^{\gamma+1}_{n_k\, a_{n_k}}$ are different.

We postpone the proof of this proposition to Section 4. We prove Proposition 2 assuming it. We need the following result (we give the proof for completeness).

Lemma 5 (Claim 4.3 in [2])**.** *Let f be a diffeomorphism having a pair of saddles p and q such that $H(p,f) = H(q,f)$. Consider a saddle r of f such that $q <_{\rm us} r <_{\rm us} p$. Then the saddles p, r, and q are in the same chain recurrent class.*

Proof: It suffices to see that for every $\varepsilon > 0$ there is a closed ε-pseudo-orbit containing q, r and p. First, as $H(p,f) = H(q,f)$ and this set is transitive, there is x_1 $\varepsilon/2$-close to $f(p)$, $x_1 \in H(p,f)$, such that $f^{n_1}(x_1)$ is $\varepsilon/2$-close to q.

Since $q <_{\rm us} r$, there is some $x_2 \in W^u(\mathcal{O}_q, f) \cap W^s(\mathcal{O}_r, f)$. Therefore there are positive numbers n_2 and m_2 such that $f^{-n_2}(x_2)$ is $\varepsilon/2$-close to $f(q)$ and $f^{m_2}(x_2)$ is $\varepsilon/2$-close to r. Similarly, $r <_{\rm us} p$ gives $x_3 \in W^u(\mathcal{O}_r, f) \cap W^s(\mathcal{O}_p, f)$ and positive numbers n_3 and m_3 such that $f^{-n_3}(x_3)$ is $\varepsilon/2$-close to $f(r)$ and $f^{m_3}(x_3)$ is $\varepsilon/2$-close to p.

The announced closed ε-pseudo-orbit containing p, r, and q is obtained concatenating the segments of orbits above:

$$p, x_1, \dots, f^{n_1-1}(x), q, f^{-n_2}(x_2), \dots, f^{m_2-1}(x_2), r, f^{-n_3}(x_3), \dots, f^{m_3-1}(x_3), p.$$

The proof of the lemma is now complete. □

We are now ready to conclude the proof of Proposition 2.

Lemma 6. *Let $\gamma \in [\alpha, \beta-1] \cap \mathbb{N}$ and q^γ_φ and $q^{\gamma+1}_\varphi$ saddles as in Lemma 4 (generically homoclinically linked, having real multipliers and indices γ and $\gamma+1$) defined for every diffeomorphism $\varphi \in \mathcal{G}_{\mathcal{U}}$. There is a dense subset $\mathcal{D}^\gamma_{\mathbb{R}}$ of $\mathcal{U}$ of diffeomorphisms ϕ having a heterodimensional cycle associated to q^γ_ϕ and $q^{\gamma+1}_\phi$.*

Proof: By Lemma 4, the diffeomorphism φ has saddles q^γ_φ and $q^{\gamma+1}_\varphi$ such that $H(q^\gamma_g, g) = H(q^{\gamma+1}_g, g)$ for every $g \in \mathcal{G}$ close to φ. By Lemma 3, there is ϕ arbitrarily close to φ with a heterodimensional cycle associated to q^γ_ϕ and $q^{\gamma+1}_\phi$. This ends the proof of the lemma. □

Consider the sequence $(a_k)_{k=1}^\infty$ in the proposition and fix $k \in \mathbb{N}$. Take $\phi \in \mathcal{D}^\gamma_{\mathbb{R}}$ and consider the heterodimensional cycle associated to q^γ_ϕ and $q^{\gamma+1}_\phi$. This cycle satisfies the hypothesis of Proposition 3. Thus, by Proposition 3 and Remark 1, there is an open set $\mathcal{V}_\phi(k)$ such that

- ϕ is in the closure of $\mathcal{V}_\phi(k)$;
- for every diffeomorphism $g \in \mathcal{V}_\phi(k)$, there is $n_g(k) > k$ such that g has $n_g(k)\, a_{n_g(k)}$ different orbits $\mathcal{O}_{r_1^\gamma(g)}, \dots, \mathcal{O}_{r^\gamma_{n_g(k)\, a_{n_g(k)}}(g)}$ of period $n_g(k)$ and index γ and $n_g(k)\, a_{n_g(k)}$ different orbits $\mathcal{O}_{r_1^{\gamma+1}(g)}, \dots, \mathcal{O}_{r^{\gamma+1}_{n_g(k)\, a_{n_g(k)}}(g)}$ of period $n_g(k)$ and index $\gamma+1$;
- the saddles verify $q_g <_{\rm us} r_i^\gamma(g) <_{\rm us} p_g$ and $q_g <_{\rm us} r_i^{\gamma+1}(g) <_{\rm us} p_g$, for all $i = 1, \dots, n_g(k)\, a_{n_g(k)}$.

For each $k \in \mathbb{N}$, consider the set

$$\mathcal{V}^\gamma(k) = \bigcup_{\phi \in \mathcal{D}^\gamma_{\mathbb{R}}} \mathcal{V}_\phi(k).$$

By construction and since $\mathcal{D}^{\varphi}_{\mathbb{R}}$ is dense in $\mathcal{U}$, the set $\mathcal{V}^\gamma(k)$ is open and dense in $\mathcal{U}$. Consider now the set

$$\mathcal{G}^\gamma(k) = \mathcal{G}_{\mathcal{U}} \cap \mathcal{V}^\gamma(k) \subset \mathcal{G}.$$

This set is residual in $\mathcal{U}$. By construction, $q_g <_{\mathrm{us}} r_i^\gamma(g) <_{\mathrm{us}} p_g$ and $q_g <_{\mathrm{us}} r_i^{\gamma+1}(g) <_{\mathrm{us}} p_g$, for all $1 \le i \le n_g(k)\, a_{n_g(k)}$. Thus Lemma 5 implies that chain recurrence classes of $p_g, r_i^\gamma(g), r_i^{\gamma+1}(g)$, and q_g coincide for all $g \in \mathcal{G}^\gamma(k) \subset \mathcal{G}$ and all $i = 1, \dots, n_g(k)\, a_{n_g(k)}$. By (G2), the homoclinic classes of these saddles coincide for all $g \in \mathcal{G}^\gamma(k)$ and all $i = 1, \dots, n_g(k)\, a_{n_g(k)}$. Since $n_g(k) \ge k$, the set $\mathcal{G}^\gamma(k)$ verifies the conclusion in the Proposition 2. □

4. **Generation of saddles at heterodimensional cycles.** In this section, we prove Proposition 3. We need the following preparatory result which is essentially proved in [2]:

Proposition 4. *Let f be a diffeomorphism having a pair of periodic saddles p_f and q_f of indices γ and $\gamma+1$, respectively, and with real multipliers. Assume that the diffeomorphism f has a heterodimensional cycle associated to p_f and q_f.*

Then for every C^1-neighborhood $\mathcal{U}$ of f there are constants $k_0 \in \mathbb{N}$ and $C > 0$ such that for every pair of natural numbers l_0 and m_0 there are $l > l_0$ and $m > m_0$ and a diffeomorphism $g_{\ell,m} \in \mathcal{U}$ having a periodic point $r_{\ell,m}$ such that:

1. *the period $\pi(r_{\ell,m})$ of $r_{\ell,m}$ is $\ell\, \pi(p_f) + m\, \pi(q_f) + k_0$, where $\pi(p_f)$ and $\pi(q_f)$ are the periods of p_f and q_f;*
2. *the point $r_{\ell,m}$ is partially hyperbolic, there is a $Df^{\pi(r_{\ell,m})}(r_{\ell,m})$-invariant dominated* splitting $T_{r_{\ell,m}}M = E^{ss} \oplus E^c \oplus E^{uu}$ such that E^{ss} and E^{uu} are uniformly hyperbolic (contracting and expanding, respectively),* $\dim E^c = 1$, *and* $\dim E^{uu} = \gamma$;
3. *the eigenvalue $\lambda_c(r_{\ell,m})$ of $Df^{\pi(r_{\ell,m})}(r_{\ell,m})$ corresponding to the central direction E^c satisfies*
$$1/C < |\lambda_c(r_{\ell,m})| < C;$$
4. $q_g <_{\mathrm{us}} r_{\ell,m} <_{\mathrm{us}} p_g$.

Sketch of the proof of Proposition 4: This proposition follows from the arguments in [2, Theorem 3.2]. Note, in [2] the index is the dimension of the stable bundle. For completeness, we outline the main steps and ingredients of the proof of this proposition. For details, see [2, 12].

By hypothesis, the saddles p_f and q_f have real eigenvalues, thus there are eigenvalues λ_c of $Df^{\pi(p_f)}(p_f)$ and β_c of $Df^{\pi(q_f)}(q_f)$ such that $\lambda < \lambda_c < 1$ for every contracting eigenvalue λ of $Df^{\pi(p_f)}(p_f)$ and $\beta > \beta_c > 1$ for every expanding eigenvalue β of $Df^{\pi(q_f)}(q_f)$. The eigenvalues λ_c and β_c are the *central eigenvalues of the cycle.*

---

*A Df-invariant splitting $E \oplus F$ of TM over an f-invariant set Λ is *dominated* if the fibers of the bundles have constant dimension and there are a metric $||\cdot||$ and a natural number $n \in \mathbb{N}$ such that

$$||Df^n(x)_E|| \cdot ||Df^{-n}(x)_F|| < \frac{1}{2}, \quad \text{for all } x \in \Lambda.$$

For splittings with three bundles $E \oplus F \oplus G$, domination means that the splittings $(E \oplus F) \oplus G$ and $E \oplus (F \oplus G)$ are both dominated.

PSfrag replacements

q_f
p_f
x
y
$\mathfrak{T}_{(p,q)}$
f^n
f^{-n}
$\mathfrak{T}_{(q,p)}$
f^m
f^{-m}
U_p
U_q

FIGURE 2. An affine heterodimensional cycle

The fact that the saddles p_f and q_f have real multipliers also implies that there is a (unique) Df-invariant dominated splitting defined on the union of the orbits $\mathcal{O}_{p_f}$ of p_f and $\mathcal{O}_{q_f}$ of q_f,

$$T_xM = E^{ss}_x \oplus E^c_x \oplus E^{uu}_x, \qquad x \in \mathcal{O}_{p_f} \cup \mathcal{O}_{q_f},$$

where $\dim E^c_x = 1$ and $\dim E^{uu}_x = \gamma$. We let $\nu = \dim E^{ss}_x = n - \gamma - 1$.

After a C^1-perturbation of f, one gets a new heterodimensional cycle (associated to the same saddles p_f and q_f with real multipliers) and local coordinates at these saddles such that the dynamics of f in a neighborhood of the cycle is affine (this corresponds to the notion of *affine heterodimensional cycle* in [2, Section 3.1]). Let us explain this point more precisely. For that we introduce some notation. The elements are depicted in the figure.

We fix small neighborhoods U_p and U_q of the orbits of p_f and q_f and heteroclinic points $x \in W^s(p_f, f) \cap W^u(q_f, f)$ and $y \in W^u(p_f, f) \cap W^s(q_f, f)$. After a perturbation, we can assume that

- the intersection between $W^s(p_f, f)$ and $W^u(q_f, f)$ at x is transverse, and
- the intersection between $W^u(p_f, f)$ and $W^s(q_f, f)$ at y is quasi-transverse (i.e., $T_yW^u(p_f, f) + T_yW^s(q_f, f) = T_yW^u(p_f, f) \oplus T_yW^s(q_f, f)$ and this sum has dimension $n - 1$, where n is the dimension of the ambient manifold).

Then there are neighborhoods U_x of x and U_y of y and natural numbers n and m such that

$$f^n(U_x) \subset U_p, \quad f^{-n}(U_x) \subset U_q, \quad f^{-m}(U_y) \subset U_p, \quad \text{and} \qquad f^m(U_y) \subset U_q.$$

We say that $t_{(q,p)} = 2\,n$ and $t_{(p,q)} = 2\,m$ are *transition times* from U_q to U_p and from U_p to U_q, respectively. The maps

$$\mathfrak{T}_{(p,q)} = f^{t_{(p,q)}} \quad \text{and} \quad \mathfrak{T}_{(q,p)} = f^{t_{(q,p)}}$$

are *transition maps* from U_p to U_q and from U_q to U_p. These maps are defined on small neighborhoods U^-_x of $f^{-n}(x)$ and U^-_y of $f^{-m}(y)$.

Using the domination, by increasing n and m and after a small perturbation, one can assume that the transition maps preserve the dominated splitting $E^{ss} \oplus E^c \oplus E^{uu}$

defined above. More precisely, in the neighborhoods U_p and U_q the expressions of $f^{\pi(p_f)}$ and $f^{\pi(q_f)}$ are linear. Then, in these local charts, the splitting $E^{ss}\oplus E^c\oplus E^{uu}$ is of the form

$$E^{ss}=\mathbb{R}^\nu\times\{(0,0^\gamma)\},\quad E^c=\{0^\nu\}\times\mathbb{R}\times\{0^\gamma\},\quad E^{uu}=\{(0^\nu,0)\}\times\mathbb{R}^\gamma.$$

In this way, one also has that the maps

$$\mathfrak{T}_{(q,p)}=f^{t_{(q,p)}}\colon U_x^-\to U_p \qquad\text{and}\qquad \mathfrak{T}_{(p,q)}=f^{t_{(p,q)}}\colon U_y^-\to U_q$$

are affine maps preserving the splitting $E^{ss}\oplus E^c\oplus E^{uu}$. More precisely,

$$\mathfrak{T}_{(i,j)}=(T^s_{(i,j)},T_{(i,j)},T^u_{(i,j)}),\quad i,j=p,q\quad\text{or}\quad i,j=q,p,$$

where

$$T^s_{(i,j)}\colon\mathbb{R}^\nu\to\mathbb{R}^\nu,\qquad T^c_{(i,j)}\colon\mathbb{R}\to\mathbb{R},\qquad T^u_{(i,j)}\colon\mathbb{R}^\gamma\to\mathbb{R}^\gamma,$$

are affine maps such that $T^s_{(i,j)}$ is a contraction (i.e., its norm is less than one), $T^u_{(i,j)}$ is an expansion (i.e., $(T^u_{(i,j)})^{-1}$ is a contraction). We let $\tau_{(i,j)}$ the derivative of $T^c_{(i,j)}$. In fact, $T^c_{(p,q)}$ is a linear map.

The previous construction gives the dynamics of f in a neighborhood V of the cycle,

$$V=U_p\cup U_q\cup\left(\cup_{i=-n}^n f^i(U_x)\right)\cup\left(\cup_{i=-m}^m f^i(U_y)\right),$$

where we shrink (if necessary) U_p, U_q, U_x, and U_y.

The next step consists in considering the unfolding of this cycle. For this we consider a one-parameter family of transitions $(\mathfrak{T}_{(p,q),\rho})_\rho$ from p_f to q_f defined as follows

$$\mathfrak{T}_{(p,q),\rho}=\mathfrak{T}_{(p,q)}+(0^\nu,\rho,0^\gamma).$$

To each small ρ corresponds a diffeomorphism f_ρ (which is a local perturbation of f at the heteroclinic point y) such that, for every large ℓ and m, there is a small subset $U_{\ell,m}$ of U_x^- such that

$$f_\rho^{\pi_{\ell,m}}|_{U_{\ell,m}}=f^{m\,\pi(q_f)}\circ\mathfrak{T}_{(p,q),\rho}\circ f^{\ell\,\pi(p_f)}\circ\mathfrak{T}_{(q,p)},\quad \pi_{\ell,m}=\ell\,\pi(p_f)+t_{(q,p)}+m\,\pi(q_f)+t_{(p,q)}.$$

Moreover, $f_\rho^{\pi_{\ell,m}}$ maps $U_{\ell,m}$ into U_q.

Next, we find a parameter $\rho=\rho_{\ell,m}$ such that f_ρ has a periodic point of period $\pi_{\ell,m}$. Note that, by construction, in a neighborhood of the cycle, the diffeomorphism f_ρ keeps invariant the codimension one foliation generated by the sum of the strong stable and strong unstable directions (hyperplanes parallel to $\mathbb{R}^\nu\times\{0\}\times\mathbb{R}^\gamma$). Also note that f_ρ acts hyperbolically on these hyperplanes. We consider the quotient dynamics of f_ρ by this strong stable/strong unstable foliation, obtaining a one-dimensional map. Fixed points of this quotient dynamics will correspond to periodic points of the diffeomorphism f_ρ. Let us explain this point more precisely.

Suppose for simplicity that, in local coordinates,

$$x^-=f^{-n}(x)=(0^\nu,1,0^\gamma)\in U_q\quad\text{and}\quad x^+=f^n(x)=(0^\nu,-1,0^\gamma)\in U_p. \tag{1}$$

Note that, by definition of $\tau_{(p,q)}$, $T^c_{(p,q)}(z)=\tau_{(p,q)}\,z$ (in fact, the case $T^c_{(p,q)}(z)=-\tau_{(p,q)}\,z$ is simpler). Next, fix ℓ and m large and take $\rho_{\ell,m}$ such that

$$\beta^m\left(-\tau_{(p,q)}\,\lambda^\ell+\rho_{\ell,m}\right)=1,\qquad \rho_{\ell,m}=\beta^{-m}+\tau_{(p,q)}\,\lambda^\ell.$$

This choice and $T^c_{(q,p)}(1)=-1$ imply that the quotient dynamics satisfy:

$$\beta^m\circ T^c_{(p,q),\rho}\circ\lambda^\ell\circ T^c_{(q,p)}(1)=\beta^m\left(-\tau_{(p,q)}\,\lambda^\ell+\rho_{\ell,m}\right)=1.$$

Thus 1 is a fixed point of the quotient dynamics.

Since $f_{\ell,m} = f_{\rho_{\ell,m}}$ preserves the E^{ss}, E^{uu}, and E^c directions, the hyperbolicity of the directions E^{ss} and E^{uu} implies that the map

$$f^{m\,\pi(q_f)} \circ \mathfrak{T}_{(p,q),\rho_{\ell,m}} \circ f^{\ell\,\pi(p_f)} \circ \mathfrak{T}_{(q,p)}$$

has a fixed point $r_{\ell,m}$ of the form $r_{\ell,m} = (r^\nu_{\ell,m}, 1, r^\gamma_{\ell,m})$ in $U_{\ell,m}$, where $|r^\nu_{\ell,m}| \to 0$ and $|r^\gamma_{\ell,m}| \to 0$ as $\ell, m \to \infty$ (i.e., as $\rho_{\ell,m} \to 0$). The point $r_{\ell,m}$ is a periodic point of period $\pi_{\ell,m}$ of $f_{\ell,m}$.

By construction, the periodic point $r_{\ell,m}$ is uniformly expanding in the E^{uu} direction, uniformly contracting in the E^{ss} direction, and the derivative of $Df^{\pi_{\ell,m}}_{\ell,m}(r_{\ell,m})$ the central direction is

$$\lambda_c(r_{\ell,m}) = \kappa_{\ell,m} = \beta^m\, \tau_{(q,p)}\, \lambda^\ell\, \tau_{(p,q)}.$$

Note that we can choose large ℓ and m with

$$\beta^{-1} \le |\beta^m\, \tau_{(q,p)}\, \lambda^\ell\, \tau_{(p,q)}| \le \beta.$$

Taking $C = \beta$ and $k_0 = t_{(q,p)} + t_{(p,q)}$, we get the first three items in the proposition.

The last item of the proposition, $q_g <_{\rm us}< r_{\ell,m} <_{\rm us} p_g$ is exactly [2, Proposition 3.10]. Consider the points $h_{\ell,m}$ and $d_{\ell,m}$ of the $f_{\ell,m}$-orbit of $r_{\ell,m}$,

$$h_{\ell,m} = f^{-m\,\pi(q_f)}(r_{\ell,m}) = (h^\nu_{\ell,m}, \beta^{-m}, h^\gamma_{\ell,m}),$$

$$d_{\ell,m} = f^{-t_{(p,q)}}_{\ell,m}(h_{\ell,m}) = \mathfrak{T}^{-1}_{(p,q),\rho_{\ell,m}}(h_{\ell,m}) = f^{\ell\,\pi(p_f)+t_{(q,p)}}_{\ell,m}(r_{\ell,m}) = (d^\nu_{\ell,m}, \lambda^\ell, d^\gamma_{\ell,m}).$$

The key step is to observe that, by construction,

$$\Delta^s_{\ell,m} = [-1,1]^\nu \times \{(\beta^{-m}, h^\gamma_{\ell,m})\} \subset W^s(h_{\ell,m}, f_{\ell,m}) \subset U_q,$$

$$\Delta^u_{\ell,m} = \{(d^\nu_{\ell,m}, \lambda^\ell)\} \times [-1,1]^\gamma \subset W^u(d_{\ell,m}, f_{\ell,m}) \subset U_p,$$

where $h^\nu_{\ell,m} \to 0^\nu$ and $d^\gamma_{\ell,m} \to 0^\gamma$. For details, see [2, Lemma 3.11].

Noting that, in the coordinates in U_q, $\{0^\nu\} \times [-1,1]^{\gamma+1}$ is contained in the unstable manifold of the orbit of $q_f = q_{f_{\ell,m}}$, one has that $W^u(q_{f_{\ell,m}}, f_{\ell,m}) \cap \Delta^s_{\ell,m} \neq \emptyset$, thus $q_{f_{\ell,m}} <_{\rm us} r_{\ell,m}$. The relation $r_{\ell,m} <_{\rm us} p_{f_{\ell,m}}$ follows noting that, in the local coordinates in U_p, $[-1,1]^{\nu+1} \times \{0^\gamma\}$ is contained in the stable manifold of the orbit of $p_f = p_{f_{\ell,m}}$. This concludes the sketch of the proof of the proposition. □

4.1. **Proof of Proposition 3.** Using Lemma 1, we next consider a perturbation of the dynamics of $f_{\ell,m}$ along the orbit of $r_{\ell,m}$. We multiply the derivative of $f_{\ell,m}$ along the central direction by a factor

$$|\kappa_{\ell,m}|^{1/\pi_{\ell,m}}$$

In this way, we have a diffeomorphism $g_{\ell,m}$ such that $r_{\ell,m}$ is a partially hyperbolic periodic point of period $\pi_{\ell,m}$ whose derivative in the central direction is the identity or minus the identity. Moreover, if $W^{ss}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m})$ is the strong stable manifold of the orbit of $r_{\ell,m}$ and $W^{uu}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m})$ is the strong unstable manifold of the orbit of $r_{\ell,m}$ we have that

$$W^u(\mathcal{O}_q, g_{\ell,m}) \pitchfork W^{ss}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m}) \neq \emptyset \quad \text{and} \quad W^{uu}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m}) \pitchfork W^s(\mathcal{O}_p, g_{\ell,m}) \neq \emptyset,$$

where $\pitchfork$ means transverse intersection. We now fix a pair of compact disks $\Upsilon^s \subset W^{ss}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m})$ and $\Upsilon^u \subset W^{uu}(\mathcal{O}_{r_{\ell,m}}, g_{\ell,m})$ such that $\Upsilon^s \pitchfork W^u(\mathcal{O}_q, g_{\ell,m}) \neq \emptyset$ and $\Upsilon^u \pitchfork W^s(\mathcal{O}_p, g_{\ell,m}) \neq \emptyset$.

Consider now the sequence $(a_k)_{k=1}^\infty$ of natural numbers in the proposition. Given now any κ, $\kappa > \pi_{\ell,m}\, a_{\pi_{\ell,m}}$, there is $\phi^\kappa_{\ell,m}$ arbitrarily close to $g_{\ell,m}$ such that $\phi^\kappa_{\ell,m}$ has

κ-saddles $r_1^{\gamma+1}, \dots, r_\kappa^{\gamma+1}$ of index $\gamma+1$ and κ-saddles $r_1^{\gamma}, \dots, r_\kappa^{\gamma}$ of index γ, all of them of period $\pi_{\ell,m}$ (or of period $2\,\pi_{\ell,m}$ if we begin the construction with minus the identity), whose orbits are arbitrarily close to the $g_{\ell,m}$-orbit of $r_{\ell,m}$. This implies that

- The stable manifold of the orbit of r_i^γ contains a disk close to Υ^s. Thus $W^u(\mathcal{O}_q, \phi^\kappa_{\ell,m}) \pitchfork W^s(\mathcal{O}_{r_i^\gamma}, \phi^k_{\ell,m}) \neq \emptyset$, and $q <_{\rm us} r_i^\gamma$.
- The unstable manifold of the orbit of r_i^γ contains a disk close to Υ^u. Thus $W^s(\mathcal{O}_p, \phi^k_{\ell,m}) \pitchfork W^u(\mathcal{O}_{r_i^\gamma}, \phi^k_{\ell,m}) \neq \emptyset$, and $r_i^\gamma <_{\rm us} p$.

A similar argument holds for the saddles $r_1^{\gamma+1}, \dots, r_\kappa^{\gamma+1}$ of index $(\gamma+1)$. Therefore, $q <_{\rm us} < r_i^{\gamma+1} <_{\rm us} p$, for all $i = 1, \dots \kappa$. This completes the proof of the proposition. □

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