MAXIMAL ENTROPY MEASURES FOR CERTAIN PARTIALLY HYPERBOLIC, DERIVED FROM ANOSOV SYSTEMS

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ABSTRACT. We show that a class of robustly transitive diffeomorphisms originally described by Mañé are intrinsically ergodic. More precisely, we obtain an open set of diffeomorphisms which fail to be uniformly hyperbolic and structurally stable, but nevertheless have the following stability with respect to their entropy.

Their topological entropy is constant and they each have a unique measure of maximal entropy with respect to which periodic orbits are equidistributed. Moreover, equipped with their respective measure of maximal entropy, these diffeomorphisms are pairwise isomorphic.

We show that the method applies to several classes of systems which are similarly derived from Anosov, i.e., produced by an isotopy from an Anosov system, namely, a mixed Mañé example and one obtained through a Hopf bifurcation.

1. INTRODUCTION

Let f be a diffeomorphism of a manifold M to itself. The diffeomorphism f is *transitive* if there exists a point $x \in M$ where

$$\mathcal{O}_f^+(x) = \{f^n(x) | n \in \mathbb{N}\}$$

is dense in M. It is *robustly transitive* [4, Ch. 7] if there exists a neighborhood $\mathcal{U}$ of f in the space $\mathrm{Diff}^1(M)$ of C^1 diffeomorphisms such that each g in $\mathcal{U}$ is transitive. Since robust transitivity is an open condition, it is an important component of the global picture of dynamical systems [23].

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The first examples of robustly transitive diffeomorphisms were transitive Anosov diffeomorphisms (see Section 2 for definitions). Nonhyperbolic robustly transitive diffeomorphisms were then constructed by Shub [26] and Mañé [18]. These examples satisfy a weaker hyperbolic condition called partial hyperbolicity (see Section 2). It is interesting to note when results for Anosov diffeomorphisms continue to hold and when the properties are very different. For instance, C^1-structural stability holds for Axiom A systems with strong transversality and no others [18]. In this paper we analyze measures of maximal entropy, and a related notion of stability, for some class of non-Anosov robustly transitive diffeomorphisms based on Mañé's example. Our method applies to some systems derived by isotopy from a C^0-close Anosov system.

To state our results we need to give some definitions. Dynamical entropies are measures of the complexity of orbit structures [6]. The topological entropy, $h_{\mathrm{top}}(f)$, considers all the orbits, the entropy $h_{\mathrm{top}}(f, S)$ of a set S considers those that start from this set, whereas the measure theoretic entropy, $h_\mu(f)$, focuses on those "relevant" to a given invariant probability measure μ. The variational principle (see for example [16, p. 181]) says that if f is a continuous self-map of a compact metrizable space and $\mathcal{M}(f)$ is the set of invariant probability measures for f, then

$$h_{\mathrm{top}}(f) = \sup_{\mu\in\mathcal{M}(f)} h_\mu(f).$$

A measure $\mu \in \mathcal{M}(f)$ such that $h_{\mathrm{top}}(f) = h_\mu(f)$ is a *measure of maximal entropy*. By a theorem of Newhouse [21] we know that C^∞ smoothness implies the existence of such measures (but finite smoothness does not, according to Misiurewicz [20]). If there is a unique measure of maximal entropy, then f is called *intrinsically ergodic*. One can then inquire about the following weakening of structural stability:

Definition 1.1. *We say $f \in \mathrm{Diff}^1(M)$ is* **intrinsically stable** *if there exists a neighborhood $\mathcal{U}$ of f such that each g in $\mathcal{U}$ has a unique measure of maximal entropy μ_g and, (g, μ_g) is a measure-preserving transformation isomorphic to (f, μ_g).*

The following property also holds for maximal entropy measures of Anosov systems:

Definition 1.2. *Let $f \in \mathrm{Diff}(M)$. Let $\epsilon > 0$ and $\mathrm{Per}_\varepsilon(n)$ be an (ε, n)-separated subset of $\{x \in M : x = f^n(x)\}$, the set of periodic points. A diffeomorphism f is said to have* **equidistributed periodic points** *with respect to a measure μ if the following holds for any small enough*

$\varepsilon > 0$:

$$\lim_{n\to\infty} \frac{1}{|\mathrm{Per}_\varepsilon(n)|} \sum_{x\in \mathrm{Per}_\varepsilon(n)} \delta_x = \mu.$$

Newhouse and Young [22] have shown that the structurally unstable examples of Abraham and Smale [1] on $\mathbb{T}^4$ are intrinsically stable (and in particular intrinsically ergodic). This also applies to the robustly transitive diffeomorphisms constructed by Shub. The present work extends this (by a less topological and more dynamical method) to several derived from Anosov systems (systems that are C^0-close to an Anosov system and obtained from a certain kind of perturbation ,see for instance [24, p. 334] for a precise definition), and in particular the robustly transitive diffeomorphisms constructed by Mañé on $\mathbb{T}^3$ whose construction will be recalled in Section 4.

Theorem 1.3. *For any $d \geq 3$, there exists a non-empty open set $\mathcal{U}$ in* $\mathrm{Diff}(\mathbb{T}^d)$ *satisfying:*

- *each $f \in \mathcal{U}$ is strongly partially hyperbolic, robustly transitive, and intrinsically stable (in particular the topological entropy is locally constant at f);*
- *each $f \in \mathcal{U}$ has equidistributed periodic points with respect to the measure of maximal entropy; and*
- *no $f \in \mathcal{U}$ is Anosov or structurally stable.*

The above result will be deduced from the following abstract theorem. Consider a homeomorphism of a compact metric space $f : X \to X$. Assume that it is expansive and has the specification property: these properties are recalled in Sec. 2. For now, let us note that they hold for all Anosov diffeomorphisms and ensure the existence of a unique measure of maximal entropy (see [8]). Let $g : X \to X$ be a continuous extension of f, i.e., there is a continuous surjective map $\pi : X \to X$ such that $f \circ \pi = \pi \circ g$.

The map π defines an equivalence relation: $y \sim_\pi z$ if and only if $\pi(y) = \pi(z)$. For $x \in X$ we denote by

$$[x] := \{y \in X : \pi(y) = \pi(x)\} = \pi^{-1}(\pi(x))$$

the equivalence class of x.

We say that a class $[x]$ is *periodic* if $\pi(x)$ is a periodic point of f. In this case, $g^m|_{[x]} : [x] \to [x]$ is a homeomorphism on the compact set $[x]$, where m is the period of $\pi(x)$. So, there exists a g^m-invariant probability measure supported on $[x]$. We pick one and denote it by

$\delta_{[x]}$. We can assume that $\delta_{[gx]} = g_* \delta_{[x]}$. Of course,

$$\frac{1}{m} \sum_{k=0}^{m-1} g_*^k \delta_{[x]}$$

is a g-invariant probability measure supported on the orbit of the periodic class $[x]$. We define $\widetilde{\operatorname{Per}}_n(g)$ as the set of equivalent classes that are fixed by g^n and set

$$\nu_n := \frac{1}{|\widetilde{\operatorname{Per}}_n(g)|} \sum_{x \in \widetilde{\operatorname{Per}}_n(g)} \delta_x, \tag{1}$$

Remark 1.4. *If a periodic class $[x]$ contains a periodic point of g with the same minimal period as $\pi(x)$, then we simply set $\delta_{[x]} = \delta_x$. This is precisely the situation in Theorem 1.3.*

Theorem 1.5. *Let $f : X \to X$ be an expansive homeomorphism of a compact metric space with the specification property and let μ be the unique measure of maximal entropy of f.*

Let $g : X \to X$ be a continuous extension of f through some continuous surjective map $\pi : X \to X$ and assume that the following two conditions are satisfied:

(H1) $h_{\text{top}}(g, [x]) = 0$ *for any* $x \in X$,
(H2) $\mu(\{\pi(x) : [x]$ *is reduced to* $\{x\}\}) = 1$.

Then, recalling (1), *the following limit*

$$\nu = \lim_{n \to \infty} \nu_n \tag{2}$$

exists. Furthermore, the measure ν is g-invariant, ergodic, and is the unique measure maximizing the entropy of g.

We will apply Theorem 1.5 to several classes that are derived from Anosov, namely:

- *a mixed Mañé* example (see Sec. 5);
- a system derived from Anosov through *a Hopf bifurcation* (see Sec. 6).

Remark 1.6. *Looking from another point of view, a preprint version of this article raised the following question:* Is every robustly transitive diffeomorphism intrinsically ergodic? intrinsically stable?

Examples of Kan [4, 15] *suggested that the answer might be negative. These maps admit two SRB measures on the boundary circles that are also measures of maximal entropy. Nevertheless, they are robustly transitive within C^1 self-maps of the compact cylinder preserving* *the boundary. Indeed,* [25] *recently announced a negative answer to the above question.*

A follow up paper [10] of JB and TF will analyze a set of robustly transitive diffeomorphisms on $\mathbb{T}^4$, based on examples of Bonatti and Viana [5] which have the weakest possible form of hyperbolicity for robustly transitive diffeomorphisms, a dominated splitting [3, 12] (see Sec. 2).

We note that Hua, Saghin, and Xia [14] have also proved local constancy of the topological entropy, e.g., in the class of partially hyperbolic diffeomorphisms C^1 close to toral automorphisms with at most one eigenvalue on the unit circle. Their method relies on certain homological data of the center foliation.

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2. Background

We now review a few facts on entropy, hyperbolicity, and partial hyperbolicity.

Let X be a compact metric space and f be a continuous self-map of X. Fix $\epsilon > 0$ and $n \in \mathbb{N}$. Let $\operatorname{cov}(n, \epsilon, f)$ be the minimum cardinality of a covering of X by (ϵ, n)-balls, i.e., sets of the form

$$\{y \in X : \; d(f^k(y), f^k(x)) < \epsilon \text{ for all } 0 \le k \le n\}.$$

The *topological entropy* is [6]

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0} (\limsup_{n \to \infty} \frac{1}{n} \log \operatorname{cov}(n, \epsilon, f)).$$

Let $Y \subset X$ and $\operatorname{cov}(n, \epsilon, f, Y)$ be the minimum cardinality of a cover of Y by (n, ϵ)-balls. Then the *topological entropy of* Y *with respect to* f is

$$h_{\text{top}}(f, Y) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \operatorname{cov}(n, \epsilon, f, Y).$$

If (X, f) and (Y, g) are continuous and compact systems and $\phi : X \to Y$ is a continuous surjection such that $\phi \circ f = g \circ \phi$, then $h_{\text{top}}(g) \le h_{\text{top}}(f)$ (f is called an *extension* of g and g is called a *factor* of f). For the definition of measure theoretic entropy refer to [16, p. 169].

A homeomorphism $f : X \to X$ is *expansive* with expansivity constant $r_0 > 0$ if for all $x, y \in X$, $\sup_{n \in \mathbb{Z}} d(f^n x, f^n y) < r_0 \implies x = y$. It has the *specification property* if for any $\varepsilon > 0$, there exists an integer

D such that for all $r \in \mathbb{N}$, $(x_1, n_1), \ldots, (x_r, n_r) \in X \times \mathbb{N}$, there exists $z \in X$ such that $d(f^{n_1+\cdots+n_s+sD+m}(z), f^m(x_{s+1})) < \varepsilon$ for all $0 \leq s < r$, all $0 \leq m < n_{s+1}$ and $f^{n_1+\cdots+n_r+rD}(z) = z$. See [8]

An invariant set Λ is *hyperbolic* for $f \in \mathrm{Diff}(M)$ if there exists an invariant splitting $T_\Lambda M = E^s \oplus E^u$ and an integer $n \geq 1$ such that Df^n uniformly contracts E^s and uniformly expands E^u: so for any point $x \in \Lambda$,

$$\begin{aligned} \|Df_x^n v\| &\leq \tfrac{1}{2}\|v\|, \text{ for } v \in E_x^s, \text{ and} \\ \|Df_x^{-n} v\| &\leq \tfrac{1}{2}\|v\|, \text{ for } v \in E_x^u. \end{aligned}$$

If $A \in \mathrm{GL}(d, \mathbb{Z})$ has no eigenvalues on the unit circle, then the induced map f_A of the d-torus is called a *hyperbolic toral automorphism*. By construction any hyperbolic toral automorphism is Anosov.

If Λ is a hyperbolic set, $x \in \Lambda$, and $\epsilon > 0$ sufficiently small, then the *local stable and unstable manifolds* at x are respectively:

$$\begin{aligned} W_\epsilon^s(x, f) &= \{y \in M \mid \text{for all } n \in \mathbb{N}, d(f^n(x), f^n(y)) \leq \epsilon\}, \text{ and} \\ W_\epsilon^u(x, f) &= \{y \in M \mid \text{for all } n \in \mathbb{N}, d(f^{-n}(x), f^{-n}(y)) \leq \epsilon\}. \end{aligned}$$

The *stable and unstable manifolds* of x are respectively:

$$\begin{aligned} W^s(x, f) &= \{y \in M \mid \lim_{n\to\infty} d(f^n(y), f^n(x)) = 0\}, \text{ and} \\ W^u(x, f) &= \{y \in M \mid \lim_{n\to\infty} d(f^{-n}(y), f^{-n}(x)) = 0\}. \end{aligned}$$

They can be obtained from the local manifolds as follows:

$$\begin{aligned} W^s(x, f) &= \textstyle\bigcup_{n\geq 0} f^{-n}\left(W_\epsilon^s(f^n(x), f)\right), \text{ and} \\ W^u(x, f) &= \textstyle\bigcup_{n\geq 0} f^{n}\left(W_\epsilon^u(f^{-n}(x), f)\right). \end{aligned}$$

For a C^r diffeomorphism the stable and unstable manifolds of a hyperbolic set are C^r injectively immersed submanifolds.

An ϵ-*chain* from a point x to a point y for a diffeomorphism f is a sequence $\{x = x_0, ..., x_n = y\}$ such that

$$d(f(x_{j-1}), x_j) < \epsilon \text{ for all } 1 \leq j \leq n.$$

A standard result that applies to Anosov diffeomorphisms is the Shadowing Theorem, see for example [24, p. 415]. Let $\{x_j\}_{j=j_1}^{j_2}$ be an ϵ-chain for f. A point y δ-*shadows* $\{x_j\}_{j=j_1}^{j_2}$ provided $d(f^j(y), x_j) < \delta$ for $j_1 \leq j \leq j_2$. We remark that there are much more general versions of the next theorem, but the following statement will be sufficient for the present work.

Theorem 2.1. *(Shadowing Theorem) If f is an Anosov diffeomorphism, then given any $\delta > 0$ sufficiently small there exists an $\epsilon > 0$*

such that if $\{x_j\}_{j=j_1}^{j_2}$ *is an* ϵ*-chain for* f*, then there is a* y *which* δ*-shadows* $\{x_j\}_{j=j_1}^{j_2}$*. If* $j_2 = -j_1 = \infty$*, then* y *is unique. If, moreover, the* ϵ*-chain is periodic, then* y *is periodic.*

A diffeomorphism $f : M \to M$ has a *dominated splitting* if there exists an invariant splitting $TM = E_1 \oplus \cdots E_k$, $k \geq 2$, (with no trivial subbundle) and an integer $l \geq 1$ such that for each $x \in M$, $i < j$, and unit vectors $u \in E_i(x)$ and $v \in E_j(x)$, one has

$$\frac{\|Df^l(x)u\|}{\|Df^l(x)v\|} < \frac{1}{2}.$$

A diffeomorphism f is *partially hyperbolic* if there is a dominated splitting $TM = E_1 \oplus \cdots \oplus E_k$ and $n \geq 1$ such that Df^n either uniformly contracts E_1 or uniformly expands E_k. We say f is *strongly partially hyperbolic* if there exists a dominated splitting $TM = E^s \oplus E^c \oplus E^u$ and $n \geq 1$ such that Df^n uniformly contracts E^s and uniformly expands E^u.

For f a strongly partially hyperbolic diffeomorphism we know there exist unique families $\mathcal{F}^u$ and $\mathcal{F}^s$ of injectively immersed submanifolds such that $\mathcal{F}^i(x)$ is tangent to E^i for $i = s, u$, and the families are invariant under f, see [13]. These are called, respectively, the unstable and stable laminations[1] of f. For the center direction, however, there are examples where there is no center lamination [28]. For a strongly partially hyperbolic diffeomorphism with a 1-dimensional center bundle it is not known if there is always a lamination tangent to the center bundle. On the other hand, a C^1 center foliation is plaque expansive and hence structurally stable (but with possible loss of regularity) [13]. Let us quote a special case of this result:

Theorem 2.2. [13, Theorems (7.1) and (7.2)] *Let* f *be a* C^1 *diffeomorphism of a compact manifold* M*. If* f *is strongly partially hyperbolic with a* C^1 *central foliation* $\mathcal{F}$*, then any* g C^1*-close to* f *has a* C^1 *central lamination* $\mathcal{G}$ *and there is a homeomorphism* $h : M \to M$ *such that for all* $x \in M$*, (i) the leaf* $\mathcal{F}_x$ *is mapped by* h *to the leaf* $\mathcal{G}_{hx}$*; (ii)* $g(\mathcal{G}_{hx}) = \mathcal{G}_{h(fx)}$.

This applies in particular to the Mañé example.

[1] A C^r foliation is a partition of the manifolds locally C^r-diffeomorphic (or homeomorphic if $r = 0$) to a partition of $\mathbb{R}^d$ into k-planes for some $0 \leq k \leq d$. A lamination is a C^0 foliation with C^1 leaves.

3. A sufficient criterion for intrinsic ergodicity

R. Bowen [8] established equidistribution of periodic points for expansive homeomorphism with the specification property. In this section we generalize this to some well-behaved extension of such systems, obtaining the abstract theorem announced in the introduction.

Proof of Theorem 1.5. Let ν be any accumulation point of the sequence ν_n. We will prove that ν is the unique measure of maximal entropy, and hence ν will be the limit of ν_n and the result will follows. We will split the proof into several lemmas.

Lemma 3.1. *In the setting of the theorem, $\pi_*\nu = \mu$.*

Proof. First, note that $\pi_*\nu_n = \mu_n$, for all $n \geq 1$. In fact, for every $A \subseteq X$ Borel, if $x \in X$ is any point, then

$$\delta_{[x]}(\pi^{-1}(A)) = \delta_{\pi(x)}(A).$$

We conclude that

$$\begin{aligned}
\pi_*\nu_n(A) &= \nu_n(\pi^{-1}(A)) \\
&= \frac{1}{|\widetilde{\mathrm{Per}}_n(g)|} \sum_{x \in \widetilde{\mathrm{Per}}_n(g)} \delta_{[x]}(\pi^{-1}(A)) \\
&= \frac{1}{|\mathrm{Per}_n(f)|} \sum_{\pi(x) \in \mathrm{Per}_n(f)} \delta_{\pi(x)}(A) \\
&= \mu_n(A).
\end{aligned}$$

From the continuity of π_* we have that

$$\pi_*\nu = \mu.$$

□

Lemma 3.2. *The measure ν is of maximal entropy, that is,*

$$h_\nu(g) = h_{\mathrm{top}}(g).$$

Proof. (g, ν) being an extension of (f, μ), $h_\mu(f) \leq h_\nu(g)$. On the other hand, Bowen's formula [6] states that

$$h_{\mathrm{top}}(g) \leq h_{\mathrm{top}}(f) + \sup_{x \in X} h_{\mathrm{top}}(g, [x]).$$

Therefore, from (H1) and the variational principle we conclude that

$$h_{\mathrm{top}}(g) \leq h_{\mathrm{top}}(f) = h_\mu(f) \leq h_\nu(g) \leq h_{\mathrm{top}}(g).$$

□

We say that A is *saturated* if $A = \pi^{-1}(\pi(A))$. In general, the *saturation* of $A \subseteq X$ is defined as $\mathrm{sat}(A) := \pi^{-1}(\pi(A))$. Note that $\nu(\mathrm{sat}(A)) = \mu(\pi(A))$.

Lemma 3.3. *For every Borel set A we have $\nu(A) = \nu(\mathrm{sat}(A))$.*

Proof. Let $\tilde{X} = \{x \in X \,:\, [x] = \{x\}\}$. From (H2) and the fact that $\pi_*\nu = \mu$ we have that $\nu(\tilde{X}) = 1$. For $A \subseteq X$ Borel, we have

$$\nu(\mathrm{sat}(A)) = \nu(\mathrm{sat}(A) \cap \tilde{X}) = \nu(A \cap \tilde{X}) = \nu(A).$$

□

Corollary 3.4. *The probability measure ν is ergodic.*

Proof. From Lemma 3.3 it follows that if P is a g-invariant subset, then

$$\nu(P) = \nu(\pi^{-1}(\pi(P)) = \mu(\pi(P)).$$

Since $\pi(P)$ is f-invariant and μ is ergodic we know that ν is ergodic. □

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Lemma 3.5. *Let η be a g-invariant probability measure and assume that η is singular with respect to ν. Then*

$$h_\eta(g) < h_{\mathrm{top}}(g).$$

Proof. Observe that it is enough to prove the lemma for η ergodic and then use the ergodic decomposition.

Let $\rho = \pi_*\eta$. The measure ρ is therefore ergodic. We claim that ρ and μ are mutually singular.

We proceed by contradiction. By ergodicity, they are equal. But assumption (H2) says that π is one-to-one over a set of full measure for μ. Hence, $\eta = \rho$ contrarily to assumption. The claim is proved.

The Ledrappier-Walter's formula [17] states that

$$h_\eta(g) \leq h_\rho(f) + \int_X h_{\mathrm{top}}(g, \pi^{-1}(x))d\rho(x).$$

and from (H1) it follows that

$$h_\eta(g) \leq h_\rho(f).$$

Bowen proved [8] that $h_\rho(f) < h_{\mathrm{top}}(f) = h_{\mathrm{top}}(g)$ and the result follows.

□

Now, we can finish the proof of Theorem 1.5. Let η be any g-invariant probability measure such that $h_\eta(g) = h_{\mathrm{top}}(g)$. We can write

$$\eta = \alpha\eta_1 + (1-\alpha)\eta_2$$

for some $\alpha \in [0,1]$ such that η_i are invariant probability measures, $\eta_1 << \nu$ and η_2 is singular with respect to ν. It follows that

$$h_{\mathrm{top}}(g) = h_\eta(g) = \alpha h_{\eta_1}(g) + (1-\alpha)h_{\eta_2}(g) \leq h_{\mathrm{top}}(g).$$

The previous lemma implies that $\alpha = 1$, that is, η is absolutely continuous with respect to ν. As ν is ergodic we have that $\eta = \nu$. This completes the proof of the theorem. □

4. Intrinsic ergodicity for Mañé's robustly transitive diffeomorphisms

Mañé's example of a robustly transitive dynamical system that is not Anosov was constructed on $\mathbb{T}^3$, but can be extended to higher dimensions.

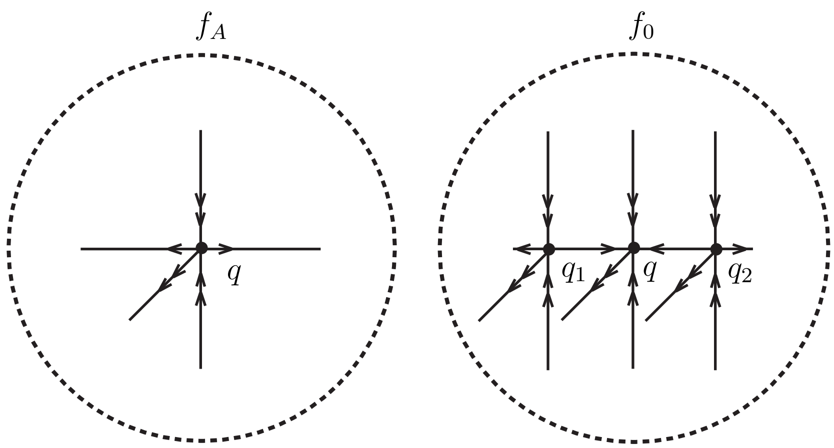

Figure 1. Mañé's construction

We fix some dimension $d \geq 3$ and let $A \in \mathrm{GL}(d,\mathbb{Z})$ be a hyperbolic toral automorphism with only one eigenvalue inside the unit circle and all eigenvalues real, positive, simple, and irrational. Let λ_s be the unique modulus less than 1 and λ_c be the smallest of the moduli greater than 1.

We denote the induced linear Anosov system on $\mathbb{T}^d$ by f_A and let $\mathcal{F}^c$ be the foliation corresponding to the eigenvalue λ_c. Locally, at each point, $\mathcal{F}^c$ is just a line segment in the direction of an eigenvector associated with λ_c. Similarly, $\mathcal{F}^s$ and $\mathcal{F}^u$ are the foliations corresponding to an eigenvalue λ_s and all the eigenvalues greater than λ_c, respectively. Since all eigenvalues are irrational, each leaf of $\mathcal{F}^s$, $\mathcal{F}^c$, and $\mathcal{F}^u$ is dense in $\mathbb{T}^d$.

Such matrices can be built for any $d \geq 3$ as companion matrices to the minimal polynomial over $\mathbb{Q}$ of a Pisot number whose algebraic conjugates are all real. Such numbers are given by Theorem 5.2.2 in [2, p. 85] (the proof implies that the conjugates are real). The moduli are then pairwise distinct by [27].

Without loss of generality, we may assume that f_A has at least two fixed points and that any unstable eigenvalue other than λ_c has modulus greater than 3 (if not, replace A by some power).

Let p and q be fixed points under the action of f_A and $\rho > 0$ be a small number to be determined below. Following the construction in [18] we define f_0 by modifying f_A in a sufficiently small domain C contained in $B_{\rho/2}(q)$ keeping invariant the foliation $\mathcal{F}^c$. So there is a neighborhood U of p such that $f_A|_U = f_0|_U$. Inside C the fixed point q undergoes a pitchfork bifurcation in the direction of the foliation $\mathcal{F}^c$. The stable index of q increases by 1, and two other saddle points with the same stable index as the initial q are created. (See Figure 1.)

The resulting diffeomorphism f_0 is strongly partially hyperbolic with a C^1 center foliation $\mathcal{F}^c$. According to [18], it is also robustly transitive (in fact topologically mixing [4, p. 184]) for $\rho > 0$ sufficiently small.

The next proposition will be helpful in the proof of Theorem 1.3.

Proposition 4.1. *(Shadowing proposition) Let f_A be an Anosov diffeomorphism of the d-torus, $d \geq 3$, as above. Let $f \in \mathrm{Diff}^1(\mathbb{T}^d)$ satisfy the following properties:*

(a) *there exist constants $\epsilon > 0$ and $\delta > 0$ such that each ϵ-chain under f_A is δ-shadowed by an orbit under f_A and 3δ is an expansive constant for f_A, (i. e. if $x, y \in \mathbb{T}^d$ and $d(f_A^n(x), f_A^n(y)) < 3\delta$ for all $n \in \mathbb{Z}$, then $x = y$), and*

(b) *each f-orbit is an ϵ-chain for f_A.*

Then the map $\pi : \mathbb{T}^d \to \mathbb{T}^d$, where $\pi(x)$ is the point in $\mathbb{T}^d$ that under the action of f_A will δ-shadow the f-orbit of x, is a semiconjugacy from f to f_A, i.e., it is a continuous and onto map with $\pi \circ f = f_A \circ \pi$.

Proof. By the shadowing theorem we know that the map π is well-defined and that $\pi(f(x)) = f_A(\pi(x))$ and $d(\pi(x), x) < \delta$. We need to see that π is continuous [26, Theorem 7.8] and surjective. It is folklore, but we provide a proof for the convenience of the reader.

To show that π is continuous we take a sequence $x_n \to x$ and show that $\pi(x_n) \to \pi(x)$. Fix $M \in \mathbb{N}$. Then there exists an $N(M) \in \mathbb{N}$ such that for each $n \geq N(M)$ we have

$$d(f^j(x_n), f^j(x)) < \delta \text{ for all } -M \leq j \leq M.$$

We then have

$$d(f_A^j(\pi(x_n)), f_A^j(\pi(x))) < 3\delta \text{ for all } -M \leq j \leq M$$

where $n \geq N(M)$. It follows that for any limit point y of the sequence $\{\pi(x_n)\}$ we have

$$d(f_A^j(y), f_A^j(\pi(x))) \leq 3\delta \text{ for all } j \in \mathbb{Z}. \tag{3}$$

Since 3δ is an expansive constant for f_A this implies that $y = \pi(x)$ and $\pi(x_n)$ converges to $\pi(x)$.

We now show that π is surjective. Assume that it is not and let $y \notin \pi(\mathbb{T}^d)$. Consider the closed ball $\overline{B} = \overline{B}(y, 3\delta)$ and the map from the ball to its boundary $r : \overline{B} \to \partial\overline{B}$ as follows: for $x \in \overline{B}, r(x)$ is intersection of the geodesic ray (starting at y and passing through $\pi(x)$) with the boundary $\partial\overline{B}$. The map is well defined since $\pi(x) \neq y$. Moreover it is continuous. On the other hand $r_{/\partial\overline{B}} : \partial\overline{B} \to \partial\overline{B}$ is isotopic to the identity (since $d(\pi(x), x) < \delta$ and the ball has radius 3δ). This contradicts Brouwer's Theorem.□

We shall also use the following (folklore) fact:

Lemma 4.2. *Let $g : \mathbb{T}^d \to \mathbb{T}^d$ be an injective continuous self-map. Let K be a compact curve such that the lengths of all its iterates, $g^n(K)$, $n \geq 0$, are bounded by a constant L. Then $h(g, K) = 0$.*

Proof of Lemma For each $n \geq 0$, there exists a subset $K(\varepsilon, n)$ of $g^n(K)$ with cardinality at most $L/\varepsilon + 1$ dividing $g^n(K)$ into curves with length at most ε. Observe that $\bigcup_{0 \leq k < n} g^{-k}K(\varepsilon, k)$ is an (n, ϵ)-cover of K with subexponential cardinality. □

Proof of Theorem 1.3 The strategy of the proof of Theorem 1.3 is to use the semiconjugacy π_g from Proposition 4.1 and to show that for each $x \in \mathbb{T}^d$ and each g C^1-close to f_0, the set $\pi_g^{-1}(x)$ is a compact interval of bounded length contained in a center leaf, and $\pi_g^{-1}(x)$ is a unique point for almost every x. These facts, together with Lemma 4.2 and Theorem 1.5 imply the result. We note that the measure of maximal entropy for f_A is Lebesgue measure, denoted μ, on $\mathbb{T}^d$.

We note that for $\rho > 0$ small enough, any diffeomorphism f that is C^1 close to the previously constructed diffeomorphism f_0, satisfies the hypothesis of Proposition 4.1.

Let $r > 0$ be an expansive constant for f_A and fix a neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of f_0 such that each $g \in \mathcal{U}$ satisfies the hypothesis of Proposition 4.1 with $0 < \epsilon < \delta < \min(r/3, \rho)$. For each $g \in \mathcal{U}$ we denote π_g as the semiconjugacy mapping g to f_A given by Proposition 4.1.

Let μ be Lebesgue measure on $\mathbb{T}^d$ and set

$$m = \mu(B(q, 3\rho)) > 0. \tag{4}$$

The above construction is such that the maximum contraction in the center direction, denoted $b(f)$, satisfies

$$\lambda_c^{1-m} b(f)^{2m} > 1 \tag{5}$$

where m is defined in (4).

Fix $\gamma > 0$ such that $(\lambda_c - \gamma)^{1-m}(b(f) - \gamma)^{2m} > 1$. Possibly by reducing $\mathcal{U}$, we may and do assume that $d_{C^1}(f_0, g) < \gamma$ and that robust transitivity holds for all $g \in \mathcal{U}$.

Fix $g \in \mathcal{U}$ and suppose that $y_1, y_2 \in \pi_g^{-1}(x)$. By construction of π_g, this implies $d(g^n(y_1), g^n(y_2)) < 2\delta$ for all $n \in \mathbb{Z}$. The normal hyperbolicity of the center lamination implies that such y_1 and y_2 must lie in the same center leaf. By the bounded curvature property, the whole segment of $\mathcal{F}^c$ between y_1 and y_2 stays within $2\delta < r$ of the orbit of y_1, hence its image by π_g stays within $\epsilon + 2\delta < r$ of the orbit of x so this interval must be contained in $\pi_g^{-1}(x)$. It follows that the set $\pi_g^{-1}(x)$ is a compact interval in a center leaf which keeps a bounded length under all iterates of g. The above lemma implies that $h(g, \pi_g^{-1}(x)) = 0$ for all $x \in \mathbb{T}^d$. Thus, (H1) of Theorem 1.5 is satisfied.

To prove that g itself is intrinsically ergodic, it is just left to prove (H2), in other words that π_g is almost everywhere one-to-one: Lebesgue almost every point in $\mathbb{T}^d$ has a unique pre-image under π_g. Since μ is ergodic for f_A we know from Birkhoff's ergodic theorem (see [24, p. 274]) that for μ-almost every $x \in \mathbb{T}^d$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_{B(q,\rho+2\delta)}(f_A^i(x)) = \mu(B(q, \rho + 2\delta)) = m. \tag{6}$$

Fix $g \in U$ and let

$$a(g) = \min_{x\in\mathbb{T}^3 - B(q,\rho)} Dg_x\mathcal{F}^c(x) \geq \lambda_c - \gamma$$

and

$$b(g) = \min_{x\in B(q,\rho)} Dg_x\mathcal{F}^c(x) \geq b(f) - \gamma.$$

So $a(g)$ measures the minimum expansion in $\mathbb{T}^d - B(q, \rho)$ in the center direction and $b(g)$ measures the maximum contraction in $B(q, \rho)$ in the center direction. We know that if $\pi_g(z) = \pi_g(y)$, then $d(z, y) < 2\delta$. So if $y \in \mathbb{T}^d - B(q, \rho + 2\delta)$, then $z \notin B(q, \rho)$ and

$$|Dg_z\mathcal{F}^c| \geq a(g) \geq \lambda_c - \gamma.$$

Fix $\sigma > 0$ such that

$$(\lambda_c - \gamma)^{1-m-\sigma}(b(f) - \gamma)^{2m+\sigma} > 1.$$

Hence, for μ-almost every $x \in \mathbb{T}^d$, there exists some $K(x) > 0$ such that, for all $z \in \pi_g^{-1}(x)$, all $k \geq 0$, and

$$\begin{aligned} |Dg_z^k \mathcal{F}^c| &\geq K(x)[a(g)^{1-m-\sigma} b(g)^{2m+\sigma}]^k \\ &\geq K(x)[(\lambda_c - \gamma)^{1-m-\sigma}(b(f) - \gamma)^{2m+\sigma}]^k \\ &\geq K(x) c^k \end{aligned}$$

with $c > 1$. As $\pi_g^{-1}(x)$ must keep a bounded length it must be a unique point for μ-almost every x. This shows that (H2) holds.

Thus Theorem 1.5 applies, proving that g has a unique measure of maximal entropy ν isomorphic to that of f. Moreover, the classes of the periodic points of g are equidistributed with respect to ν. Theorem 1.3 is proved.
□

5. A Mixed Mañé Example Derived from Anosov

We now consider further classes of examples. Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a (linear) Anosov diffeomorphism, where $n \geq 4$, such that $T\mathbb{T}^n$ admits a dominated splitting, $T\mathbb{T}^n = E^{ss} \oplus E^s \oplus E^u \oplus E^{uu}$ with $\dim E^s = \dim E^u = 1$ and the rate of contraction/expansion is $\lambda_{ss} < \lambda_s < 1 < \lambda_u < \lambda_{uu}$ where λ_{ss} is the largest modulus of an eigenvalue corresponding to E^{ss}, λ_s is the modulus of the eigenvalue corresponding to E^s, λ_u is the modulus of the eigenvalue corresponding to E^u, and λ_{uu} is the smallest modulus of an eigenvalue corresponding to E^{uu}. Indeed, take any linear Anosov map A on $\mathbb{T}^2$ with $T\mathbb{T}^2 = E^s \oplus E^u$ and then take a linear Anosov map B on $\mathbb{T}^{n-2}$ such that $T\mathbb{T}^{n-2} = E^{ss} \oplus E^{uu}$ and $\lambda_{ss} := \|B|E^{ss}\| < \|A|E^s\|$, $\lambda_{uu}^{-1} := \|B^{-1}|E^{uu}\| < \|A^{-1}|E^u\|$. Then $f := A \times B$ is a linear Anosov map on $\mathbb{T}^n$ with the required properties. Notice also, that if we set $E^c = E^s \oplus E^u$ then, f is strongly partially hyperbolic diffeomorphism, $T\mathbb{T}^n = E^{ss} \oplus E^c \oplus E^{uu}$ and $\mathbb{T}^n$ has a normally hyperbolic foliation whose leaves are tori $\mathbb{T}^2$ tangent to E^c.

Taking a power of f, if necessary, assume that f has two different fixed points p and q. Let $r > 0$ be small (to be determined later) and deform f inside $B(p, r)$ and $B(q, r)$ similar to Mañé's derived from Anosov construction: in $B(p, r)$ we perform a pitchfork perturbation along E^s and on $B(q, r)$ we perform a pitchfork bifurcation along E^u (this also can be done in such a way that the foliation by tori $\mathbb{T}^2$ tangent to E^c is preserved). In this way we obtain g that falls into Proposition 4.1. Indeed, let δ, r be such that any r-chain is δ shadowed (see

Shadowing Theorem). We may assume that g satisfies the following properties:

- g is strongly partially hyperbolic: $T\mathbb{T}^n = E^{ss} \oplus E^{cs} \oplus E^{cu} \oplus E^{uu}$ which is dominated (each subbundle dominates the previous ones by a factor $a < 1$), and $\dim E^{cs} = \dim E^{cu} = 1$. These subbundles are C^0 close to the respective ones of f;
- $\mathrm{d}_{C^0}(f, g) < r$;
- if $\mathrm{d}(x, y) < 2\delta$, then $\frac{\|Dg|E^{ci}(x)\|}{\|Dg|E^{ci}(y)\|} < a^{-1/4}$, $i = s, u$;
- $Df|E^{cs}(x)$ is uniformly contracting outside $B(p, r)$ with rate λ_s; and
- $Df|E^{cu}(x)$ is uniformly expanding outside $B(q, r)$ with rate λ_u.

Notice also the above conditions hold in a neighborhood of g (the last two, the rate expansion/contraction will be close to λ_s and λ_u respectively). Moreover, by the way we constructed g, $E^c = E^{cs} \oplus E^{cu}$ is unique integrable and normally hyperbolic and hence by Theorem 2.2 the same holds in a neighborhood of g. Moreover, this example is also robustly transitive by similar arguments as in [13], since we can take a periodic central leave that does not intersects the support of the perturbation and hence supports a transitive Anosov, and by the structural stability of the central leaves, the stable and unstable manifolds of this periodic torus are dense.

We show that g falls into the assumptions of Theorem 1.5 and hence has a unique measure of maximizing entropy. This example shares similarities with the previous one and so we will give an outline of the proof. Let

$$\sigma_1 = \sup\{\|Dg|E^{cs}(x)\| \;:\; x \in M\},$$
$$\sigma_2 = \inf\{\|Dg|E^{cu}(x)\| \;:\; x \in M\},$$

and $m \in \mathbb{N}$ be such that $\sigma_1 \lambda_s^m < 1$ and $\sigma_2 \lambda_u^m > 1$. Let $\rho > 0$ satisfy

$$\mu(M \setminus B(j, \rho)) \geq 1 - 1/2m, \; j = p, q$$

where μ is the Bowen measure of f, where the *Bowen measure* is the unique measure of maximal entropy for the Anosov diffeomorphism (see for example [16, p. 618]). We assume that $2\delta < \rho/2$. Let $\pi : \mathbb{T}^n \to \mathbb{T}^n$ be the semicongugacy (from Proposition 4.1 $d(\pi, id) < \delta$). Notice that E^{cs} and E^{cu} are one-dimensional and hence they are integrable. We shall denote by $W^{cs}(x)$ a leaf (maximal integral curve) of E^{cs} containing x. We do not claim that the curve is unique. We also denote by $W^{cs}_\gamma(x)$ the arc in $W^{cs}(x)$ of size 2γ with x in the middle. Analogously we define W^{cu}. As mentioned above the foliation by $\mathbb{T}^2$ tangent to E^c is preserved by g. We denote by W^c the leaves of the foliation tangent to $E^c = E^{cs} \oplus E^{cu}$. Let J be a segment tangent to E^{cs}, we say that E^{cs}

is uniquely integrable through J if any maximal integral curve of E^{cs} through any point of J must contain J. Analogously for E^{cu}.

Lemma 5.1. *Let $x \in \mathbb{T}^n$ be any point. Then, one and only one of the following hold:*

(1) $\pi^{-1}(x)$ *consists of a single point.*
(2) $\pi^{-1}(x)$ *is a segment tangent to E^{cs} of length less than 2δ.*
(3) $\pi^{-1}(x)$ *is a segment tangent to E^{cu} of length less than 2δ.*
(4) $\pi^{-1}(x)$ *is a square tangent to $E^{cs} \oplus E^{cu}$ such that:*
- *for each $y \in \pi^{-1}(x)$ we have that $W^{cs}_\gamma(y) \cap \pi^{-1}(x)$ is a central stable segment that we denote by $J^{cs}(y)$ and E^{cs} is uniquely integrable through $J^{cs}(y)$. Similar for E^{cu}.*
- *If y and z are in $\pi^{-1}(x)$ then, $\emptyset \neq J^{cs}(y) \cap J^{cu}(z) \in \pi^{-1}(x)$.*

Proof. Assume that $\pi^{-1}(x)$ is not trivial, and let $y, z \in \pi^{-1}(x)$ be two different points. By the normal hyperbolicity, as in the previous example, we conclude that $y \in W^c(z)$. And also, if $z \in W^{cs}_\gamma(y)$ then $[y,z]^{cs} \subset h^{-1}(x)$. This means that $W^{cs}_\gamma(y) \cap h^{-1}(x)$ is a segment, say $J^{cs}(y)$ whose length remains bounded in the future and in the past and, by the domination in $E^{cs} \oplus E^{cu}$ we conclude that E^{cs} is uniquely integrable through $J^{cs}(y)$. Similar if $z \in W^{cu}_\gamma(y)$.

Assume also that neither (2) nor (3) hold. Consider local central integral curves $W^{cs}_\gamma(y)$ and $W^{cu}_\gamma(z)$ and call w the point of intersection. Although they may not have rate of expansion or contraction, a similar argument can be done so that $\pi(w) = \pi(z) = \pi(y)$. Therefore, $\{w\} = J^{cs}(y) \cap J^{cu}(z) \in \pi^{-1}(x)$. □

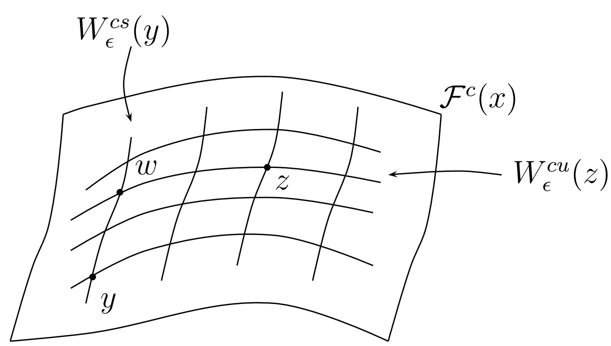

FIGURE 2

Corollary 5.2. *Conditions* (H1) *and* (H2) *are satisfied for g.*

Proof. We need only to check (H1) in case (4) above. By the above notations, we observe that

$$g(J^{cs}(y)) = J^{cs}(g(y)) \text{ and}$$
$$g(J^{cu}(y)) = J^{cu}(g(y)).$$

Therefore, the product structure is invariant and it is not difficult to see that the maximal cardinality of a (n, ϵ)-separated set in the equivalent class has at most polynomial growth. Indeed, by domination we conclude that $\|Dg^n|E^{cs}(y)\| \leq \left(a^{1/2}\right)^n$ for every n large enough, and in fact the same holds for any $w \in J^{cu}(y)$. Hence, the length of $g^n(J^{cs}(w))$ will decrease exponentially fast. We remark that a similar property holds in the past: the cu segments are contracted exponentially fast. Let $\epsilon > 0$ be given and n_0 be such that $\left(a^{1/2}\right)^{n_0} 2\delta < \epsilon/2$. Then, for $n \geq n_0$ it is not difficult to see that we can cover $g^n(\pi^{-1}(x))$ with at most $4\delta/\epsilon + 1$ balls of radius ϵ. Therefore, we have an (n, ϵ) cover of $\pi^{-1}(x)$ with polynomial growth with n. Thus, $h_{\text{top}}(g, [y]) = 0$. □

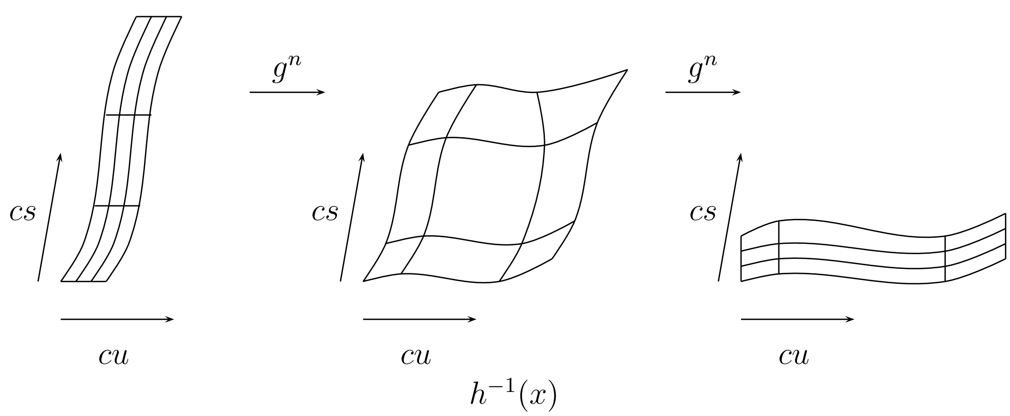

Figure 3

Finally, for condition (H2), a similar proof can be done as for the first Mañé example.

6. A Derived From Anosov Through Hopf's Bifurcation

The examples we have studied so far are robustly transitive. The next one is not. Recall that the classical Derived from Anosov map on the two-torus is obtained from an Anosov map performing a deformation on a fixed point that goes through a pitch fork bifurcation. The result is an Axiom A map, whose nonwandering set consist of a repeller fixed point and a non trivial hyperbolic attractor. In the torus $\mathbb{T}^3$ we can do a similar construction where the fixed point goes through a Hopf bifurcation instead. The result is not Axiom A. Indeed, the nonwandering set consists of a repeller fixed point and a *nonhyperbolic* transitive attractor (due to the existence of an invariant circle -corresponding to the Hopf bifurcation- inside the attractor), see [11].

We will study a particular example, the one treated in [19]. Explicit formulas and details can be found there. This example can be obtained from a linear Anosov through a (non generic) Hopf's bifurcation. In particular we do not cover the examples in [11]. Although we believe that these examples are intrinsically ergodic and that this also follows from our methods, a sharper analysis should be done (due to the possible existence of hyperbolic periodic points in the invariant circle that appears after a generic Hopf bifurcation).

Let $f_A : \mathbb{T}^3 \to \mathbb{T}^3$ be the linear Anosov diffeomorphism induced by the matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & a^2-1 & a \\ 0 & a & 1 \end{pmatrix}$$

where $a \in \mathbb{Z} - \{0\}$. As is explained in [19], this matrix is hyperbolic and has only one real eigenvalue λ_u which is bigger than 1. The other eigenvalues are complex of modulus $\lambda_s < 1$. So, f_A is a linear Anosov diffeomorphism on $\mathbb{T}^3$ and $T\mathbb{T}^3 = E^s_A \oplus E^{uu}_A$ with $\dim E^s = 2$. Consider $p = 0$ which is a fixed point of f_A. In a local coordinates and with respect to the decomposition $E^s_A \oplus E^{uu}_A$ the map f_A looks like $f_A(x, y) = (\lambda_s Rx, \lambda_u y)$ where R is a rotation. We deform f_A inside a small ball $B(p, r)$ to obtain a diffeomorphism $g : \mathbb{T}^3 \to \mathbb{T}^3$ so that in local coordinates around p (and with respect to the decomposition $E^s_A \oplus E^{uu}_A$) the map g is

$$g(x, y) = ((1 - \chi(y))\lambda_s Rx + \chi(y)\phi(\|x\|)Rx, \lambda_u y)$$

where χ is a bump function and ϕ is a smooth non-increasing function with $\phi(t) = \lambda_s$ if $t \geq 1$ and $\lambda_u > \phi(t) = \alpha > 1$ for $t \leq 0$. We see that for g the fixed point p is a repeller, surrounded by an invariant circle S (at $y = 0$ there exists a unique $c > 0$ such that $\|x\| = c$ where $\phi(c) = 1$). Indeed, it is not difficult to see that g can be viewed as the result of a Hopf bifurcation at p.

Notice also that E^s_A is invariant under Dg. Indeed, g is partially hyperbolic, $T\mathbb{T}^3 = E^{cs} \oplus E^{uu}$ where $E^{cs} = E^s_A$. We denote by W^{cs} the central stable foliation tangent to E^{cs} and by W^{uu} the (strong) unstable foliation. We also denote by $W^u(p)$ the basin of repulsion of

p and by $W^u_{\text{loc}}(p)$ the local basin of repulsion. The main properties of g are the following:

- p is a repeller for g;
- g is partially hyperbolic $T\mathbb{T}^3 = E^{cs} \oplus E^{uu}$;
- Dg uniformly contracts E^{cs} outside $B(p, r)$;
- $\|Dg|_{E^{cs}(x)}\| \leq 1$ for any $x \notin W^u_{\text{loc}}(p)$;
- $\mathrm{d}_{C^0}(g, f) < r$; and
- $E^{cs} = E^s_A$ is uniquely integrable.

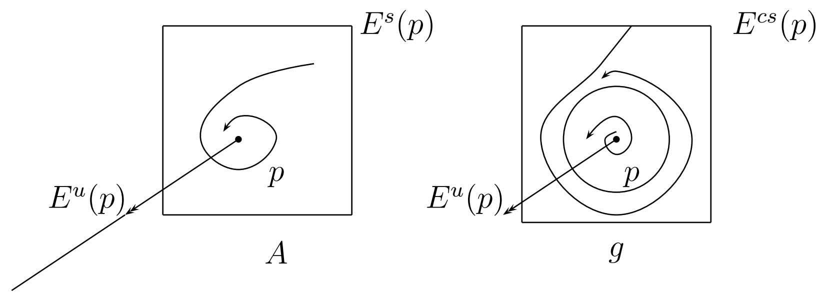

FIGURE 4

We will prove that a diffeomorphism g satisfying the above conditions has a unique measure of maximal entropy. Denote by h the semiconjugation between f and g. It is not difficult to see that h is injective on each $W^{uu}(y, g)$ for any y and moreover, $h(W^{uu}(y, g)) = W^{uu}(h(y), f_A)$. Also, we claim that $h(W^{cs}(y, g)) = W^s(h(y), f_A)$. To see this, it is better to lift g and h to the universal cover $\mathbb{R}^3$. Denote the lifts by G and H. Thus, $H \circ G = A \circ H$ and $H - Id$ is bounded. Let $z \in W^{cs}(y, G)$ and assume that $H(z) \notin W^s(H(y), A) = H(y) + E^s_A$. Notice that the distance between $G^n(y)$ and $G^n(z)$ may grow at most with exponential rate $\alpha = \sup \|Dg|_{E^{cs}}\|$. On the other hand, if $H(z) \notin W^s(H(y), A)$ then, the distance between $A^n(H(y))$ and $A^n(H(z))$ will grow with exponential rate λ_u. Since $A^n(H(y)) = H(G^n(y))$ and $A^n(H(z)) = H(G^n(z))$ we get to a contradiction. This proves our claim. In particular, it follows that an equivalent class must be contained in a central stable manifold.

Since $h(p) = p$ we have

$$W^{cs}(p) = W^s(p, f_A) = W^s(h(p), f_A) = h(W^{cs}(p))$$

and we get that the circle $S \subset W^{cs}(p)$ attracts every point in $W^{cs}(p)$ but p. Denote by D the closed disk in $W^{cs}(p)$ bounded by S. We have

$$[p] = h^{-1}(p) = \{z : d(g^n(z), p) < Cr \ \forall\, n \in \mathbb{Z}\} = D.$$

Lemma 6.1. *If* $x \notin W^{cs}(p)$*, then* $\operatorname{diam}(g^n[x]) \to 0$.

Proof. Since $x \notin W^{cs}(p)$ there are infinitely many $n \geq 0$ such that $g^n(x) \notin B(p,r)$ (and we may assume without loss of generality that $g^n(x) \notin W^u_{\text{loc}}(p)$). Therefore, $\|Dg^n|_{E^{cs}(x)}\| \to 0$ and the same holds for any $y \in [x]$ (and uniformly on y.) The conclusion follows. □

Corollary 6.2. *Conditions* (H1) and (H2) *hold.*

Proof. As we said before, the class $[p]$ is the closed disc D, with p a repeller and the boundary S attracts everything on the disk but p. Therefore, $h_{\text{top}}(g,[p]) = 0$. If $[x] \subset W^{cs}(p)$ and $[x] \neq [p]$ then the class $[x]$ is attracted by the invariant circle and so $h_{\text{top}}(g,[x]) = 0$. Now, if $[x]$ is not a subset of $W^{cs}(p)$, then $\operatorname{diam}(g^n[x]) \to 0$ and, therefore, for any ϵ and any n large enough the cardinality of any (n,ϵ)-separated set in $[x]$ is bounded, and hence $h_{\text{top}}(g,[x]) = 0$. We have proved that (H1) holds.

Condition $(H2)$ can be proved with similar methods as in the previous examples. □

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