

Dynamical Systems: Investigations into Chaos

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What is dynamical systems?

Roughly it is the study of an evolving system.

Examples:

- Planetary orbits (celestial mechanics)
- Biorhythms
- Ecosystems
- Stock market
- Iterated functions
- Flows on vector fields

Stability of the universe



In 1887, King Oscar II of Sweden offered a prize to the solution of the following question. Prize was about 1 1/2 years of salary.

Question: Is the solar system stable?

Stability means that a slight change in position or velocity won't change an orbit (much).

Henri Poincaré

The person who won the prize (and only applicant for the award) was Henri Poincaré.



Henri Poincaré (1857-1912) wrote over 400 books and papers on almost every aspect of mathematics. He can be considered the father of dynamical systems. He looked at systems not from the quantitative, but qualitative point of view.

Problem with solution

In checking the work in Poincaré's paper there was a mistake found. He was told to buy back every copy of the journal containing his original work, solve the problem, and then publish the new result at his expense.

He did this at twice the cost of the original result. The next paper explained why he couldn't solve the 3-body problem. What Poincaré had found was what we now call chaos.

3-body problem

Poincaré research was titled *On the problem of three bodies and the equations of dynamics* (translated), 1890, 270 pages.

It turns out we can easily describe the motion of particles in the universe if there are only two particles. Once we add a third particle things become complicated.

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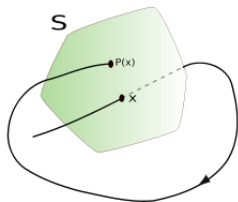
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He looked at the case where one body is massless. Think of a universe containing the sun, earth, and small asteroid. What he saw was “celestial chaos”.

For the first time someone looked at the solution of an autonomous (time-independent) differential equation and considered the geometry of the set of its trajectories.

Poincaré section

Poincaré realized he could simplify the analysis by looking at the cross section of a flow. Then a three dimensional problem turns into looking at iterations of map on the plane.



Iteration of a map

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Think of this as state of your system after one time interval. We assume that where we go after one interval only depends on where we are and not at what time we are there.

So if we start with a point $x \in \mathbb{R}^n$ and apply f we see that $f(x)$ is the value after one time interval. After two intervals we have $f(f(x)) = f^2(x)$. After n time intervals we have $f^n(x)$.

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The **forward orbit of x for f** is

$$\mathcal{O}^+(x) = \{f^n(x) \mid n \in \mathbb{N}\}.$$

If f is invertible the **orbit of x for f** is

$$\mathcal{O}(x) = \{f^n(x) \mid n \in \mathbb{Z}\}.$$

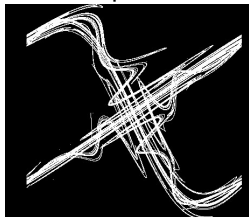
Example of iteration

Here is a simple example let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and
 $f(x, y) = (2x + y, x + y)$.

Start with the point $(x, y) = (1, 3)$. Then $f(1, 3) = (5, 4)$ and
 $f^2(1, 3) = f(5, 4) = (14, 9)$.

Homoclinic tangles and the beginning of chaos

What Poincaré found by looking at the cross sections was complicated orbit structure now called a homoclinic tangle. Not the nice periodic orbits that were expected.



Furthermore, this kind of orbit structure is quite “common” and will persist as parameters change a small amount (the flow changes a small amount).

Legacy of Poincaré

Poincaré's work on dynamical systems was largely forgotten until the 1960's.

Some work done between 1890-1960's, but not much. In the 1960's there was an explosion of research in dynamical systems for various aspects of mathematics and other areas.

Lorenz

You may not know the name Edward Lorenz (1917-2008), but you know his work. He coined the term the butterfly effect.

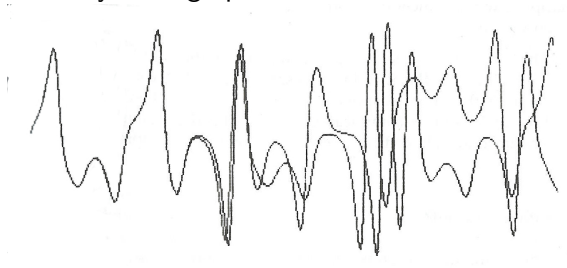


Lorenz was a meteorologist working at MIT. In 1963 he wrote a simple system of differential equations as a model for the weather.

$$\begin{aligned}x' &= -10x + 10y \\y' &= 28x - y - xz \\z' &= -\frac{8}{3}z + xy\end{aligned}$$

Chaos in the weather

Lorenz had a primitive computer and to check his work would rerun the second half of the previous day at the beginning of the next day. The graph below shows what he found.

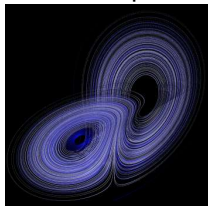


Lorenz attractor

When Lorenz went to publish his work meteorologists thought it was math and mathematicians thought it was meteorology. Eventually, mathematicians looked at it. There are now almost 700 publications, in mathematics journals, whose title contain the term “Lorenz equation.”

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What mathematicians found is that most orbits are attracted to the figure above. Now called the Lorenz attractor. This attractor is still not well understood.

The state of weather prediction

Some colleagues work on this area. One of the main problems is lack of reliable data, but even then it is unlikely to have reliable data more than 5-7 days ahead of time.

Question: What does it mean when the weather says there is a 70% chance of rain?

The logistic equation

One example of a dynamical system is the population of some kind of species. This effects us in many aspects.

A simple model of a population is given by the logistic equation.

$$f_{\mu}(x) = \mu x(1 - x) \text{ where } \mu \in (1, \infty)$$

This is just a parabola with roots at 0 and 1 and vertex at $1/2$. Iterating this function for different values of μ is very interesting.

Fixed and periodic points

A **fixed point** for a map f is point p such that $f(p) = p$. A **periodic point** is a point p such that $f^n(p) = p$ for some $n \in \mathbb{N}$.

A periodic point is called **attracting** if all nearby orbits get attracted to the periodic orbit. A periodic point is called **repelling** if all nearby orbits get “pushed away” from it.

Fixed and periodic points

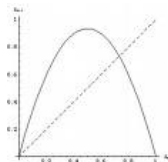
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For a population a fixed point corresponds to a steady population. A periodic point means the population fluctuates between a certain finite number of values.

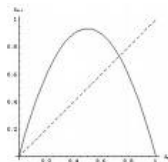
Fixed points and periodic points for the logistic map

For μ the two fixed points are 0 and another point called p_μ .



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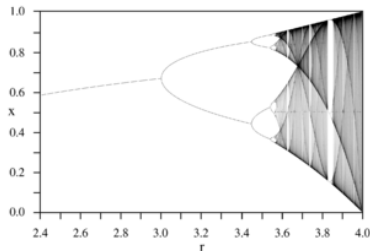
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For $\mu \in (1, 3)$ all orbits except 0 are attracted to p_μ . For $\mu > 3$ the point p_μ is repelling and there is an attracting period 2 orbit for μ between 3 and about 3.4.

Bifurcation diagrams

A **bifurcation diagram** shows the possible long-term values (fixed points or periodic orbits) of a system as a function of a parameter in the system. So the bifurcation diagram for the logistic map looks like the following. We see that between 3 and 4 we go from a very stable situation to a chaotic one.



Devaney's definition of chaos

A dynamical system (X, f) is **chaotic** if

- 1 periodic points for f are dense in X
- 2 there exists a point whose forward orbit is dense in X
- 3 the system has sensitive dependence on initial conditions (butterfly effect)

Example: The logistic map is chaotic for $\mu = 4$ and $X = [0, 1]$.

1960's dynamical systems

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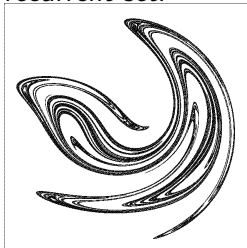
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In the west much of the work was done on the **topology of solutions**. Studying the structure of solutions. What was possible for the types of different orbits and how to they orbits behave.

In the east much of the work was done on the **probabilistic properties of solutions**. What they found was even if a system seems random in the long run there can be a nice distribution.

Topological properties

The topologists were interested in **recurrent points**. Recurrent points have orbits that come back “close” to the point. The topologists looked at the set of recurrent points to see what structure this set has. The lorenz attractor is an example of a recurrent set.



The idea of this is that orbits will tend to recurrent ones as $t \rightarrow \infty$ so the recurrent ones will tell you about the qualitative behavior of the orbits.

Probabilistic approach

Much of this work started in Moscow. Mathematicians noticed that even though the value of a function may appear random for a dynamical system the average may exist.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f^i(x)$$

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Notice that on a day to day average the stock market seems completely random, but over time looks like rather “steady”

Hyperbolic dynamical systems

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In this case one looks at iterating maps that are differentiable. The derivative gives a nice linear approximation. This helps with the local analysis.

The hyperbolicity says that the eigenvalues of the derivative matrix do not have modulus one. This allows the orbit structure to be persistent (perturbing the matrix doesn't change the action qualitatively). Meaning that if you change the map slightly the orbit structure remains the same. This then can lead to **persistent chaos**.

Beyond hyperbolicity

Work on hyperbolicity was at the forefront of dynamical systems through the 70's and early 80's. For the last twenty years researchers have been looking at the harder case of non-hyperbolic dynamical systems, but nearly hyperbolic.