Entropy - the master invariant

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Outline

Dynamical Systems

Dynamical systems is the study of the long-term behavior in systems that evolve in time with a known evolution rule.

Often the possible states of these systems can be described by observable quantities. The space of possible states is called the phase space. These quantities are often constrained, so the phase space M is often a manifold.

Discrete dynamical systems

For continuous time systems the evolution is given by a differential equation.

For discrete systems the time evolution is given by a function, $f : M \to M$. So if $x \in M$, then at one time unit later x will go to $f(x)$. At n time units we have x goes to $f^n(x)$, where $f^n(x)$ is the composition of n copies of f.

Topological conjugacy

For $f : X \to X$ and $g : Y \to Y$ a semiconjugacy from (Y, g) to (X, f) is a continuous surjective map π such that

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & X \end{array}$$

If π is injective, then (Y, g) and (X, f) are topological conjugate.

Topological conjugacy provides the most natural notion of equivalence.

Invariants

A natural problem is to find invariants for topological conjugacy. So two systems that are topologically conjugate would have the same invariants.

Finding a complete finite list of invariants is usually impossible. The "master" invariant is called entropy. The idea of entropy can be traced back to Kolmogorov (and Shannon).

Outline

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Examples

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Kolmogorov

Kolmogorov had one of the biggest impacts on dynamical systems even though his published works are under 35 pages.

In 1958 Kolmogorov used ideas of Shannon from information theory and defined the complexity of a measure-preserving transformation by the measure theoretic entropy.

Measure theoretic entropy

Dynamical entropy = growth rate of the number of orbits

If $f : X \to X$ is a measure-preserving transformation with respect to a probability measure, μ, then the measure theoretic entropy, $h_\mu(f)$, measures the exponential growth of "relevant" orbits under iteration.

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Remark 2: A positive value means the system is "chaotic" and the larger the number the more complexity. This also can tell us about the structure of the system.

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Variational Principle: If f is a continuous action of a compact metrizable space to itself and $\mathcal{M}(f)$ is the set of all invariant probability measures for f, then

$$h_{\text{top}}(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f).$$

How do we measure the number of orbits?

Let (X, d) be a compact metric space and $f : X \to X$. Then for $x, y \in X$ we define

$$d_n(x, y) = \max_{0 \le i \le n} (d(f^i(x), f^i(y)).$$

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$$\begin{aligned} &d(1/8, 1/4) = 1/8, \\ &d_1(1/8, 1/4) = 1/4, \\ &d_2(1/8, 1/4) = 1/2, \text{ and} \\ &d_n(1/8, 1/4) = 1/2 \text{ for all } n \ge 2. \end{aligned}$$

Definition of topological entropy

Fix $\epsilon > 0$ and $n \in \mathbb{N}$. A set $A \subset X$ is (n, ϵ)-separated provided $d_n(x, y) > \epsilon$ for all $x, y \in A$. Let $\mathrm{sep}(n, \epsilon, f)$ be the maximum cardinality of an (n, ϵ)-separated set for f. The topological entropy is

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Remark 3: Using a single number the topological entropy encapsulates a great deal of the complexity of the orbit structure for the system.

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Let $X = S^1$ and $f(x) = x + \alpha \mod 1$ where $\alpha \in \mathbb{R}$. This is simply a rotation by the amount α.

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Remark: By the variational principle all invariant measures have measure theoretic entropy of zero.

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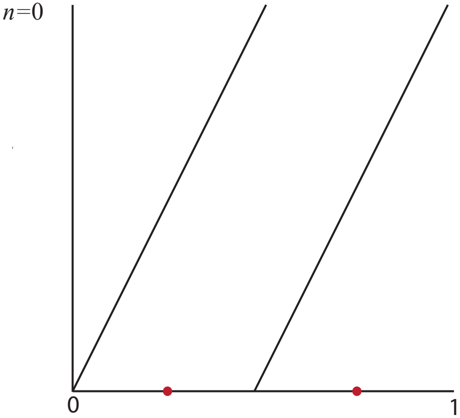

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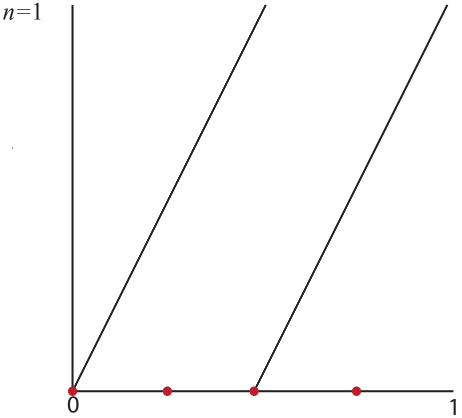

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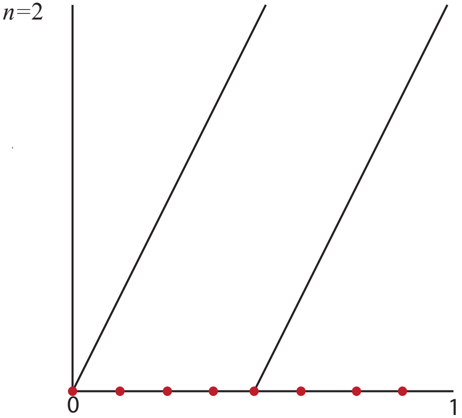

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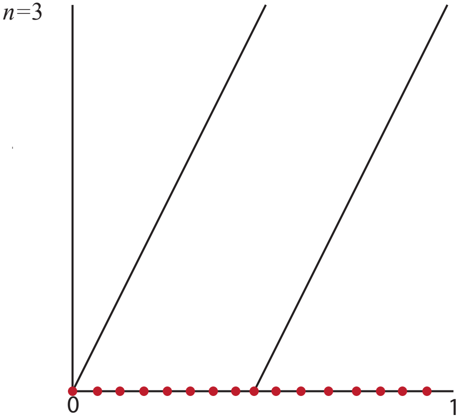

Shifts

Let $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ (infinite sequences with alphabet $\{0,1\}$.
For $s, t \in \Sigma_m^+$ let

$$d(s,t) = \sum_0^{\infty} \frac{\delta(s_k, t_k)}{2^k}$$

where

$$\delta(i,j) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

This map shifts all of the symbols one unit, so let $t \in \Sigma_m^+$ where $t = t_0 t_1 t_2 \ldots.$ Then

$$\sigma(t) = s = t_1 t_2 \ldots.$$

Entropy for shifts

$\text{sep}(1/2, 1) = 4$ and a set consists of sequences that start with

$$00, 01, 10, \text{ and } 11$$

and the same in the next positions. To find points that are $1/2$ apart for $d_1(\cdot, \cdot)$ we need to look at first 3 spots (8 of these).

So the growth of distinguishable orbits (cardinality of the separated sets) is the number of words of length n. In this case is 2^n. So $h_{\text{top}}(\sigma) = \log 2$.

Remark: It is not hard to find measures of maximal entropy for such a system by weighting each symbol with a measure of $1/m$. (Bernoulli)

Hyperbolic toral automorphisms

$A \in \mathrm{GL}_n(\mathbb{Z})$ with $|\det(A)| = 1$. This induces automorphism, f_A, of $\mathbb{T}^n$. f_A is a hyperbolic toral automorphism if no eigenvalue of A is on the unit circle.

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To see growth rate of orbits we need to only look at unstable direction. For B we see that $h_{\mathrm{top}}(f_B) = \log \lambda_1$.

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Variational principle: topological entropy = supremum of measure theoretic entropies.

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Remark 3: Usually measures of maximal entropy are singular with respect to Lebesgue measure.

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- In many cases a measure is not unique, but the space of these measures is compact and convex in the weak* topology. When the measure is unique it can be hard to show.
- In some cases the number is finite and this may be true for "most" systems, but has not been proven.

Entropy for hyperbolic systems

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- geometry
- studying structural stability

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Definition: For f a diffeomorphism of a manifold M to itself. A compact set Λ such that $f(\Lambda) = \Lambda$ is a hyperbolic set if the tangent bundle $T_\Lambda M = \mathbb{E}^s \oplus \mathbb{E}^u$ splits into continuous invariant subbundles where $\mathbb{E}^s$ is uniformly contracting and $\mathbb{E}^u$ is uniformly expanding under Df.

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Remark: For hyperbolic sets there is always a measure of maximal entropy (often unique) and the periodic points are equidistributed.

Beyond hyperbolicity

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Weakened version: A compact set Λ such that $f(\Lambda) = \Lambda$ is a partially hyperbolic set if the tangent bundle $T_\Lambda M = \mathbb{E}^s \oplus \mathbb{E}^c \oplus \mathbb{E}^u$ splits into continuous invariant subbundles such that $\mathbb{E}^s$ is uniformly contracting $\mathbb{E}^u$ is uniformly expanding and $\mathbb{E}^c$ is not contracted more than $\mathbb{E}^s$ or expanded more than $\mathbb{E}^u$.

Entropy for 1-dimensional center

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Remark 1: The idea is that 1-dimensional dynamics are well understood so we can reduce to examining the center direction.

Remark 2: In joint work with Díaz, Pacifico, and Vieitez we believe we can extend this result to the case where the center bundle dynamically splits into 1-dimensional subbundles. This is still in progress.

Entropy structure

Question: How does the measure theoretic entropy of the invariant measures converge to the topological entropy?

This is called the entropy structure of the system and tells us a great deal about the system. (So we examine the function $h : \mathcal{M}(f) \to [0, \infty)$ defined by $\mu \mapsto h_\mu(f)$.)

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Theorem 2 (Díaz and F., submitted) All diffeomorphisms with 1-dimensional center direction have a "well-behaved" entropy structure.

Theorem 3 (Díaz and F., submitted) Generically among C^1 diffeomorphisms with 2-dimensional center the entropy structure "behaves badly."

Robust transitivity

One weakening of hyperbolicity is called robust transitivity. A diffeomorphism f from a manifold to itself is transitive if there is a point with a dense forward orbit. A diffeomorphism is robustly transitive if all nearby diffeomorphisms are transitive.

Remark: All robustly transitive diffeomorphisms have a weak form of hyperbolicity

Entropy for robustly transitive diffeomorphisms

Theorem 4 (Buzzi, F., Sambarino, and Vásquez, submitted)
Certain classes of robustly transitive diffeomorphisms with 1-dimensional center direction have a unique measure of maximal entropy that is the lift of a measure from a hyperbolic system.

Entropy for robustly transitive diffeomorphisms

Theorem 4 (Buzzi, F., Sambarino, and Vásquez, submitted) Certain classes of robustly transitive diffeomorphisms with 1-dimensional center direction have a unique measure of maximal entropy that is the lift of a measure from a hyperbolic system.

Theorem 5 (Buzzi and F., in progress) Certain classes of robustly transitive systems that are not partially hyperbolic have a unique measure of maximal entropy that is the lift of a measure from a hyperbolic system.

Remark: This is the first time that a unique measure of maximal entropy has been established for systems with this weak form of hyperbolicity. Hard part is that there is not a 1-dimensional center direction.

Open questions

Question 1: Under what conditions does a unique measure of maximal entropy exist?

Question 2: For robustly transitive systems is there always a finite number of measures of maximal entropy?

Question 3: What more can be said about the entropy structure for C^r systems where $r \geq 2$?