Measures of maximal entropy for certain classes of robustly transitive diffeomorphisms

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International Conference on Nonlinear and Stochastic Dynamics
2009

Notions of entropy

In dynamical systems there are different notions of entropy. The 2 most common are:

Topological entropy: Counts the growth of orbits seen by a small scale.

Measure theoretic entropy: Counts the growth of orbits that are "relevant" to an invariant probability measure.

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Variational Principle: Let $f : X \to X$ be continuous and X be a compact metric space. Then

$$h_{\mathrm{top}}(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f)$$

where $\mathcal{M}(f)$ is the set of invariant probability measures.

Comments on the variational principle

- Misiurewicz ('73) gave examples of diffeomorphisms where no invariant probability measure achieves the maximum.
- 1 class where there is a unique measure of maximum entropy is transitive Anosov diffeomorphisms. Bowen ('75) shows this measure is given by

  $$\mu = \lim_{n\to\infty} \frac{1}{\#\mathrm{Fix}(f^n)} \sum_{x\in \mathrm{Fix}(f^n)} \delta_x.$$

  So the periodic points are equidistributed. (Margulis proved a similar result for transitive Anosov flows.)

Robustly transitive diffeomorphisms

Question: What happens when we enlarge the class of transitive Anosov diffeomorphisms?

Definition: A diffeomorphism f is **transitive** if there exists a point with a forward dense orbit. (i.e. - $\mathcal{O}^+(x) = \{f^n(x) \mid n \geq 0\}$ is dense in the manifold for some x)

Definition: A diffeomorphism f is **robustly transitive** if there exists a neighborhood $\mathcal{U}$ of f in the set of diffeomorphisms such each $g \in \mathcal{U}$ is transitive.

Note: From structural stability of hyperbolic sets we know that transitive Anosov diffeomorphisms are robustly transitive.

Non-Anosov robustly transitive diffeomorphisms

The 1^{st} examples of non-Anosov robustly transitive diffeomorphisms were due to Shub and Mañé.

Question: What can be said about measures of maximal entropy of robustly transitive diffeomorphisms? Does there exist such a measure? Is it unique? Are there only finitely many ergodic measures of maximal entropy?

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Theorem

(Newhouse, Young, '83) The robustly transitive diffeomorphisms constructed by Shub on the 4-torus have unique measures of maximal entropy that are absolutely continuous with respect to Lebesgue measure.

$1^{\rm st}$ main result

Definition: A diffeomorphism $f \in \mathrm{Diff}(M)$ is **intrinsically stable** if there exists a neighborhood $\mathcal{U}$ of f such that each $g \in \mathcal{U}$ has a unique measure of maximal entropy, μ_g, and all the μ_g are measure theoretically isomorphic. (So topological entropy is constant.)

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Theorem

(Buzzi, F., Sambarino, Vásquez) For all $d \geq 3$ there exists a nonempty open set $\mathcal{U}$ in $\mathrm{Diff}(\mathbb{T}^d)$ such that

1. *each f in $\mathcal{U}$ is strongly partially hyperbolic ($T\mathbb{T}^d = E^s \oplus E^c \oplus E^u$), robustly transitive, and intrinsically stable;*
2. *no $f \in \mathcal{U}$ is Anosov;*
3. *each $f \in \mathcal{U}$ has equidistributed periodic points.*

Mañé's construction

Take a hyperbolic toral automorphism $A \in \mathrm{GL}(d, \mathbb{Z})$ with

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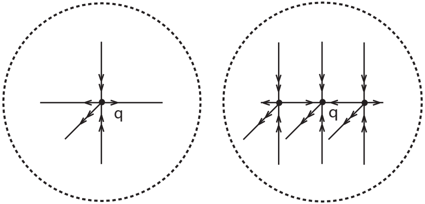
We perturb the action on $\mathbb{T}^n$ in a neighborhood about a fixed point preserving the foliation by the weakest unstable eigenvalue. The new diffeomorphism f is partially hyperbolic $T\mathbb{T}^n = E^s \oplus E^c \oplus E^u$.



Idea of the proof

Comment: Idea is that we have done something drastically topologically, but this occurs in a small neighborhood, and in 1-dimension. (**Note:** in 1-dimension (center direction) the entropy is easy to compute.) So in the measure theoretic viewpoint the change from the Anosov setting should be negligible.

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The proof proceeds as follows:

- Show there exists a neighborhood $\mathcal{U}$ of f such that for each $g \in \mathcal{U}$ there is a semi-conjugacy, π_g, from g to f_A.
- Use π_g and result of Bowen to show entropy is constant
- Use Lebesgue (measure of maximal entropy for f_A) and if ν is measure of maximal entropy for g, then $(\pi_g)_*\nu = \mu$.
- Show for Lebesgue almost every point that $\#(\pi_g(x)^{-1}) = 1$ so ν is unique.

Comments on the proof

1. To show the map π_g we make f a sufficiently small C^0 perturbation so each g orbit is shadowed by a unique point under f_A.
2. For each point we show $\pi_g(x)^{-1}$ is an interval of bounded length (could be a point).
3. This implies that $h_{\mathrm{top}}(g, \pi_g(x)^{-1}) = 0$.

2^{nd} main result

Theorem

(Buzzi, F.) For $d = 4$ there exists a nonempty open set $\mathcal{U}$ in $\mathrm{Diff}(\mathbb{T}^4)$ such that:

- *each $g \in \mathcal{U}$ is robustly transitive and intrinsically stable, and*
- *no $g \in \mathcal{U}$ is partially hyperoblic.*

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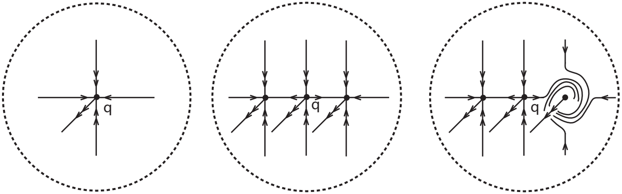
Remark:

- Novelty is that we lose the 1-dimensional center and partial hyperbolicity.
- Set $\mathcal{U}$ is based on construction of Bonatti and Viana.

Bonatti-Viana construction

Let $A \in \mathrm{GL}(4, \mathbb{Z})$ such that $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3 < \lambda_4$ and f_A has at least 3 fixed points (p, q, and r).

C^0 peturb, as before, in the first step, but then make one more perturbation (as below). Perform a similar perturbation, in the unstable direction around r.



Dominated splitting

For the new map f there is a splitting $T\mathbb{T}^4 = E^{cs} \oplus E^{cu}$ that is invariant under Df and such that there exists $\lambda > 1$ such that $\|Dfv_i\| \geq \lambda \|Dfv_j\|$ for all $v_i \in E^{cu}$ a unit vector and $v_j \in E^{cs}$ a unit vector. Such a splitting is called a **dominated splitting**.

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Remark:

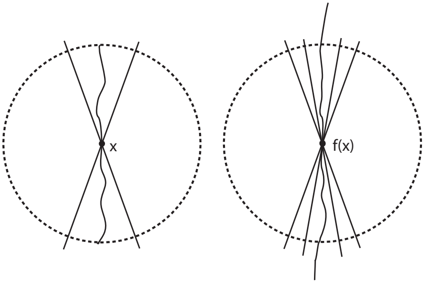
- A dominated splitting is a weak form of hyperbolicity. It says that if E^{cs} is expanding, then E^{cu} is expanding more, and if E^{cu} is contracting, then E^{cs} is expanding more.
- A dominated splitting is a robust property.

Outline of proof

- Show for a neighborhood $\mathcal{U}$ of f there such that for each $g \in \mathcal{U}$ there is a semi-conjugacy π_g to f_A.
- Show there exist foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ for E^{cs} and E^{cu}, respectively.
- Show any measure of maximal entropy is pull-back of Lebesgue as before, or concentrated at q or r.
- Then show using the foliations that measures concentrated near q and r have entropy that is too small.

Existence of foliations

We first show $\exists\ R > 0$ such that $\forall\ x \in \mathbb{T}^4\ \exists\ B^{cu}(x, R)$ contained in α-center unstable conefield and is overflowing $(f(B^{cu}(x, R) \supset B^{cu}(f(x), R))$. Since conefields are uniformly contracted we see there exists $D^{cu}(x, R)$ tangent to E^{cu}. To see $D^{cu}(x, R)$ is unique we define term (**macroscopic domination**) similar to normally hyperbolic used by Hirsch, Pugh, and Shub to show existence of foliations.



Entropy estimates

1. Show that for any measure of maximal entropy that almost every point has $\pi_g(x)^{-1}$ contained in a leaf of a foliation.
2. Show that pre-image of a point may have positive, but small, entropy.
3. Show that if a measure of maximal entropy is concentrated near a fixed point we can look at Lyapunov exponents and see that the sum in each foliation is too small to be entropy of the system.

Further questions

1. In dimension 3 we know that robust transitivity implies partial hyperbolicity. Can this be helpful in showing there is always a unique measure of maximal entropy?
2. Does there exist a robustly transitive diffeomorphism with 2 or more measures of maximal entropy? (Kan gives an example for a manifold with boundary)
3. What about studying general equilibrium states?