Symbolic extensions for partially hyperbolic diffeomorphisms

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Workshop on Partial Hyperbolicity

Entropy

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Measure theoretic entropy on the other hand measures the growth rate of orbitz "relevant" to an invariant Borel probability measure, μ, for a system (X, T). Denoted $h_\mu(T)$.

Example: If δ is a point mass, then the only "relevant" orbit is the fixed point - in this case the measure theoretic entropy is zero.

Entropy function

Let $\mathcal{M}(X, T)$ be the set of invariant Borel probability measures for (X, T). The entropy function is the map $h : \mathcal{M}(X, T) \to \mathbb{R}$ defined by $h(\mu) = h_\mu(T)$.

Theorem(Variational principle) If X is a compact metric space and T is continuous (our standing assumptions from now on), then

$$h_{\mathrm{top}}(T) = \sup_{\mu \in \mathcal{M}(X,T)} h_\mu(T).$$

Entropy structures

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- Each h_k is upper semicontinuous on $\mathcal{M}(X, T)$
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The lack of uniform convergence of an entropy structure represents how entropy emerges within the system at finer and finer scales and at different points in the system.

Symbolic extensions

The main tool in constructing these entropy structures is symbolic extensions.

Let (Σ_n, σ) be the full shift. A subshift, (Y, S), is a closed shift invariant subset of the full shift.

A symbolic extension of (X, T) is a subshift (Y, S) and a continuous surjective map $\pi : Y \to X$ such that $\pi \circ S = T \circ \pi$. (Factor map)

Note: The shift (Y, S) need not be a subshift of finite type and π need not be finite-to-one. (So different than what is often done in hyperbolic dynamics and Markov partitions.)

Symbolic extension entropy function

For (Y, S) a symbolic extension of (X, T) the extension entropy function is $h^{\pi}_{\mathrm{ext}} : \mathcal{M}(X, T) \to \mathbb{R}$ defined by

$$h^{\pi}_{\mathrm{ext}}(\mu) := \sup\{h_{\nu}(S) \,:\, \pi(\nu) = \mu\}.$$

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The symbolic extension entropy of (X, T) is

$$h_{\mathrm{sex}}(X, T) := \inf\{h_{\mathrm{top}}(Y, S) \,:\, (Y, S) \text{ is a symb. ext. of } (X, T)\}$$

The difference $h_{\mathrm{sex}}(T) - h_{\mathrm{top}}(T)$ represents complexity that is “hidden” in the multi-scale structure of the system. (The system has complexity at a local scale)

Local Entropy

One tool to study symbolic extensions go back to ideas of Bowen and Misiurewicz from the 70's.

(We now assume T is invertible to simplify the arguments.) A Bowen ball of size ϵ at x is

$$\Gamma_\epsilon(x) = \{y \in X \,:\, d(T^n(x), T^n(y) < \epsilon, \forall n \in \mathbb{Z}\}$$

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- For a hyperbolic system (Axiom A) $\Gamma_\epsilon(x) = \{x\}$ for ϵ suff. small.
- For partially hyperbolic systems we have $\Gamma_\epsilon(x)$ contained in local center manifolds for ϵ suff. small.

Entropy expansive and asymptotically expansive

Let $h^*_T(\epsilon) = \sup_{x \in X} h_{\text{top}}(T, \Gamma_\epsilon(x))$. A system is entropy expansive if $\exists\ c > 0$ such that $h^*_T(\epsilon) = 0\ \forall\ \epsilon \in (0, c)$.

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A system is asymptotically expansive if $\lim_{\epsilon \to 0} h^*_T(\epsilon) = 0$.

Note: So there may be "hidden" entropy at any arbitrarily small scale, but the amount of hidden entropy is going to zero.

Implications of asymptotically expansive systems

- Boyle, Fiebig, and Fiebig proved that any asymptotically expansive system has nice symbolic extensions, called principal symbolic extensions - an extension given by a factor map which preserves entropy for every invariant measure.

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- These properties also imply the existence of equilibrium states.

Examples

1. In the hyperbolic case we know that

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for all ϵ suff. small. So symbolic extensions exist. When hyperbolicity is relaxed this may not be the case. (A priori doesn't say a symb. ext. may not exist)

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2. Buzzi (97) proved that C^∞ diffeomorphisms of a compact manifold are asymptotically expansive. But there are C^1 examples that are not. (As above this isn't enough to say there does not exist a symb. ext.)

Weak forms of hyperbolicity

Questions:

- What happens when we weaken the hyperbolicity? For instance if a diffeomorphism is partially hyperbolic? or nonuniformly hyperbolic?
- How does the answer depend on the regularity?

Partially hyperbolic 1-dimensional dominated center subbundles

Theorem: (Díaz, F, Pacifico, Vieitez, preprint)Let $f : M \to M$ be a partially hyperbolic manifold of a compact manifold with a dominated splitting

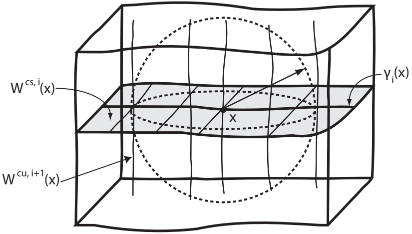
$$TM = E^s \oplus E^{c_1} \oplus \cdots \oplus E^{c_k} \oplus E^u$$

where each E^{c_i} is 1-dimensional. Then (M, f) is entropy expansive (and so has a very nice symb. ext.)

Note:
1. We actually can allow E^s and E^u to either or both be empty. Also f need only be C^1.
2. We prove a similar result for certain homoclinic classes among other sets.

Outline of proof

We show there is a uniform scale such that for every $x \in M$ there is a center curve, $\gamma_i(x)$, tangent to E^{c_i} (where i depends on x) such that $\Gamma_\epsilon(x) \subset \gamma_i(x)$. Since locally in $W^{cs,i}$ the direction E^{c_i} could act like a central curve and the other directions act locally like a stable manifold, by domination, and $W^{cu,i+1}$ act like an unstable manifold



Outline of proof -part 2

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- Under iteration the length of $\gamma_i(x)$ is bounded. Otherwise, $\Gamma_\epsilon(x)$ is a point. So folklore fact says $\Gamma_\epsilon(x)$ has entropy zero.

Condition for no symbolic extension

Downarowicz, Newhouse ('05) give a condition so no symbolic extensions exist. They need

- a sequence of partitions $\{\alpha_k\}$ (essential sequence: i.e. diameters go to zero and boundaries have measure zero) to exist and
- such that on arbitrarily small scale we see entropy greater than some constant $c > 0$, but not with respect to the partition.

So not only is there "hidden" entropy, but the amount of entropy missed with the multi scale analysis is at least c.

No symbolic extension for certain partially hyperboic sets

Theorem:(Díaz, F.) If U is an open set of partially hyperbolic diffeomorphisms, satisfying certain conditions, and with a center bundle of dimension at least two, then there is a C^1 residual set $\mathcal{R}$ in U such that each diffeomorphism in $\mathcal{R}$ has no symbolic extension.

Outline of the proof of Theorem

- Explain that the existence of center indecomposable bundle of dimension at least 2 generates persisitent homoclinic tangencies.

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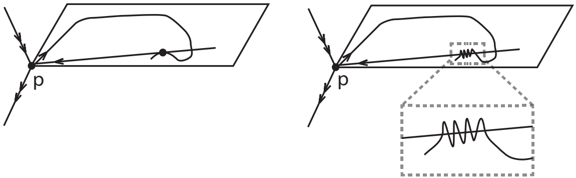
Note: Uses argument from Downarowicz and Newhouse: Show C^1 generically on surfaces conservative diffeomorphisms are Anosov or no symbolic extensions. Cataln and Tahzibi recently extended this argument to higher dimensions for sympletic diffeomorphisms.

Outline of the proof of Theorem - figure

We show for certain robustly transitive diffeos we have a periodic point p as below.



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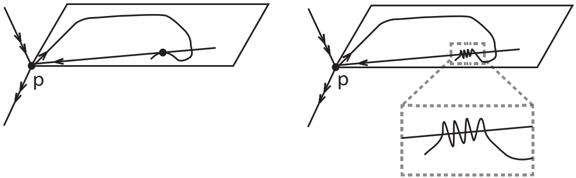

Now on a smaller scale there is a periodic point near the generalized horseshoe where we can perform the same type of perturbation.

C^2 setting

Notice the perturbations used are C^1 small but not C^2 small. (This is the case in all known examples of diffeomorphisms where no symbolic extensions are shown to exist.)

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Conjecture (Downarowicz and Newhouse, '05) Every C^r diffeomorphism of a compact space has a symbolic extension for $r \geq 2$ and

$$h_{\mathrm{sex}}(T) \leq h_{\mathrm{top}}(T) + \frac{R(t)}{r-1}$$

where $R(T)$ is the global average expansion rate (or $\lim_{n\to\infty} \frac{1}{n}\mathrm{Lip}(T^n)$ where $\mathrm{Lip}(T^n)$ is the Lipschitz constant of T^n.

Solutions to conjecture

1. Downarowicz and Mass ('09) established the conjecture for C^r maps (not necessarily diffeomorphisms) of the interval or circle.

2. Burguet ('10) established the conjecture for surface diffeomorphisms.

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2. Burguet ('10) established the conjecture for surface diffeomorphisms.

Theorem: (Burguet, F., in preparation) Let $f : M \to M$ be a C^2 partially hyperbolic diffeomorphism with 2-dimensional center bundle and with certain bunching conditions, then f has a symbolic extension.

Outline of proof

- Technique much different than argument for 1-dimensional center. We use estimates to show bound on sex entropy.
- Need local C^2 center manifolds (that is "bunching" condition on exponents)
- Use combinatorial argument to obtain estimates on growth in Bowen balls -$\Gamma_\epsilon(x)$

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Note: There is only one point where 2-dimensions is needed, but this is a key estimate.

Estimation theorem

Theorem: (Downarowicz and Maass, 09) Let (X, T) be topological system of finite entropy and $r > 1$. Let g_0 be U.S.C. on $\mathcal{M}(X, T)$ and greater than h for ergodic measures (and satisfy additional estimate with regard to what is called the Newhouse local entropy), then

$$h_{\mathrm{sex}}(\mu) \leq h_\mu(T) + \frac{\bar{g_0}(\mu)}{r-1}$$

where $\bar{g_0}$ is a (harmonic) extension of g_0 to all of $\mathcal{M}(X, T)$ and $h_{\mathrm{sex}}(\mu) := \inf h^{\pi}_{\mathrm{ext}}(\mu)$.

Estimation for 2-dim partially hyperbolic

We define $g_0 = 2\min(\chi_c^+, -\chi_c^-)(\mu)$ for μ ergodic where these are the Lyapunov exponents of μ in the center direction. We only need to look at the case where there is one positive and one negative exponent. Other cases end up being trivial.

Theorem: (Burguet, F.) Let $f \in \mathrm{Diff}^2(M)$ be partially hyperbolic with 2-dimensional center and bunching conditions and $\mu \in \mathcal{M}(X, f)$. Then

$$h_{\mathrm{sex}}(\mu) \leq h_\mu(f) + 2\min(\overline{\chi_c^+}, \overline{-\chi_c^-})(\mu).$$

In particular,

$$h_{\mathrm{sex}}(f) \leq h_{\mathrm{top}}(f) + 2\limsup_{|n|\to\infty} \frac{1}{|n|}\log^+ \|D_x f^n|_{E^c}\|.$$

Outline of estimation theorem

To prove the theorem we make estimates similar to Yomdin theory.

- We take the Bowen ball around a point and reparametrize.
- Use the Lyapunov exponents to obtain bounds on the number of reparametrizations that are needed as n grows.
- This is the point where 2-dimensions comes into play.