Quasi-Anosov diffeomorphisms of 3-manifolds

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Outline

Basic question

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Problem

Let M be a closed smooth submanifold of N, where $\dim(M) = 3$, that is a hyperbolic set for some diffeomorphism f of N. We want to classify geometry of M and dynamics of M under the action of f.

Remark

This is joint work with J. Rodriguez-Hertz.

Quasi-Anosov diffeomorphisms

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1. f *is QAD.*
2. f *is Axiom A with no cycles and for all* $x \in M$ *there is a quasi-transverse intersection for the stable and unstable manifolds (*$T_x W^s(x) \cap T_x W^u(x) = \{0\}$*).*

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4. *f is robustly expansive.*

Robustly expansive

Definition

A diffeomorphism is expansive if there exists some $\alpha > 0$ *such that* $\sup_{n\in\mathbb{Z}} d(f^n(x), f^n(y)) \leq \alpha$ *implies* $x = y$*.*

One reason to study QAD's and the manifolds that support QAD's is the hope that this will help one see restrictions expansive diffeomorphisms impose on manifolds.

Restatement of question

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Does QAD imply Anosov?

Franks and Robinson ('76) provide 3-dimensional example of QAD that is not Anosov.

Remark

In 2-dimension all QAD are Anosov. Since QAD are Axiom A with quasi-transverse intersections. So we have splittings that are all $1-1$ *and no where tangent.*

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Corollary

If M is a hyperbolic manifold for $f \in \mathrm{Diff}^1(N)$, $f|_M$ is Anosov if and only if M is locally maximal.

Outline

Franks, Robinson example

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -6 & 5 \end{bmatrix}$$

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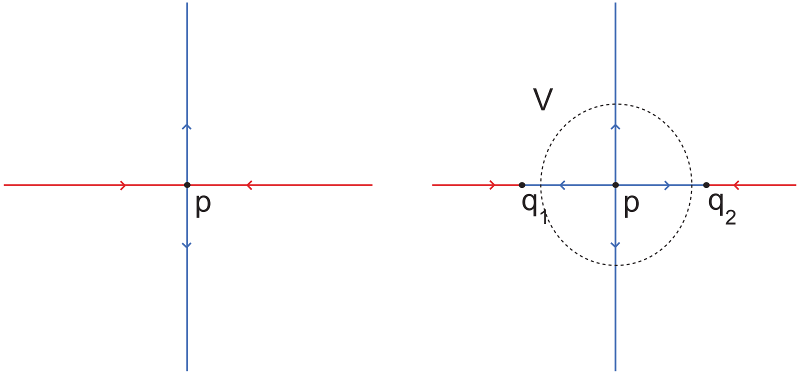
2. Let $f_2 = f_1^{-1}$ be diffeomorphism on $\mathbb{T}^3$. So there exists a repeller Λ_2 and 1-dimensional unstable foliation for Λ_2.
3. Take out neighborhood of p in each and perform surgery to attach. Adjusting so there is a quasi-transverse intersection.

Visualization of FR example - part 1



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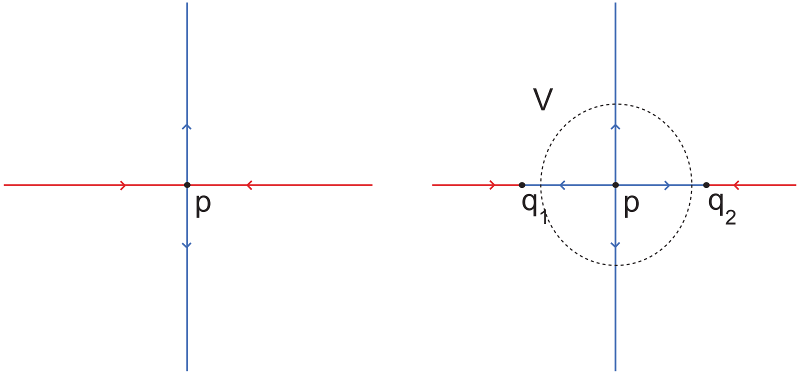
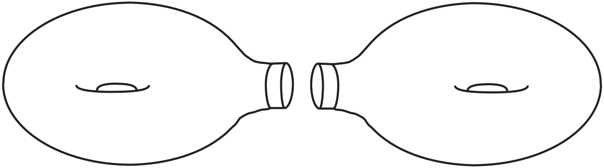
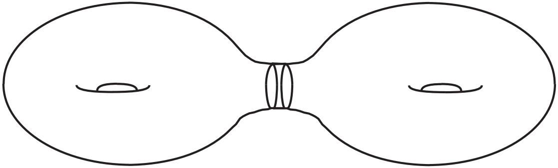


$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(M - V)$ is a codimension-one hyperbolic attractor

Visualization of FR example - part 2





Outline

First Main Result

Theorem

(F., J. Rodriguez-Hertz) Let f be a QAD of M where M is a closed orientable 3-manifold. Then the prime decomposition of M is the connected sum of k 3-tori possibly with handles. In case M is non-orientable, then the tori are quotiented by involutions.

Second Main Result

Theorem

(F., J. Rodriguez-Hertz) Let f be a QAD of a closed 3-manifold M. Then

1. *The non-wandering set $\Omega(f)$ of f consists of a finite number of codimension-one expanding attractors, codimension-one shrinking repellers and hyperbolic periodic points.*
2. *For each attractor Λ in $\Omega(f)$, there exist a hyperbolic toral automorphism A with stable index one, a finite set Q of A-periodic points, and a linear involution θ of $\mathbb{T}^3$ fixing Q such that the restriction of f to its basin of attraction $W^s(\Lambda)$ is topologically conjugate to a DA-diffeomorphism f_Q^A on the punctured torus $\mathbb{T}^3 - Q$ quotiented by θ. In case M is an orientable manifold, θ is the identity map. An analogous result holds for the repellers of $\Omega(f)$.*

Outline of arguments

1. From results of Plykin ('80) we know that the basin of attraction for a codimension-one attractor is homeomorphic to a torus minus a finite set of points. (**Note**: Using these results of Plykin the first theorem is a consequence of the second.)

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2. Using this structure we show that if Λ is a codimension-one attractor and Λ_0 is a basic set with stable dimension one where $W^u(\Lambda_0) \cap W^s(\Lambda) \neq \emptyset$, then Λ_0 is a periodic orbit.

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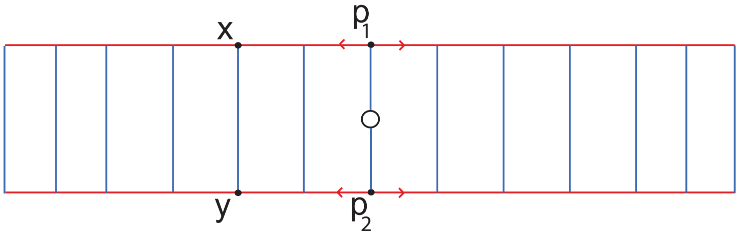
3. This implies that, if the diffeomorphism is not Anosov, all attractors and repellers of 3-dimensional QAD are codimension-one and all other basic sets consist of periodic orbits.

Proof of step 2

Definition

A point p in a codimension-one attractor Λ is a boundary point if there is a component of $W^s(p) - \{p\}$ that does not intersect Λ.

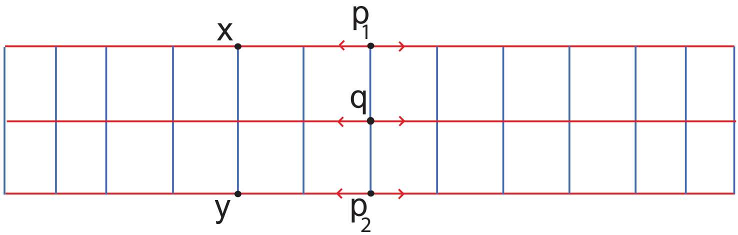
Boundary points can be shown to be periodic and occur in pairs of the same period such that if p_1 and p_2 are paired periodic points and if $x \in W^u(p_1)$, then there exists a point $y \in W^u(p_2) \cap W^s(x)$ and arc $(x, y)_s$ in $W^s(x)$ such that $(x, y)_s \cap \Lambda = \emptyset$.



Proof of step 2 (cont)

We show there if Λ_0 is codimension-one basic set such that $W^u(\Lambda_0) \cap W^s(\Lambda)$, then there is a periodic point q in Λ_0 and paired boundary points p_1 and p_2 such that $W^u(q)$ intersects an arc.

We then get a fundamental domain of $W^u(q)$ in $W^s(\Lambda)$. Which implies that Λ_0 is periodic orbit.



Additional Results

We construct an example where $\Omega(f)$ consists of codimension-one attractors and repellers and some periodic orbits (modification of Franks and Robinson).

Proposition

If $f : M \to M$ is QAD of 3 manifold and $M \neq \mathbb{T}^3$, (not Anosov), then f is approximated by QAD that are Ω-conjugate to f, but not topologically conjugate to f.

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Theorem

(F., J. Rodriguez-Hertz) If $f : M \to M$ is QAD of 3-manifold and f is partially hyperbolic, then f is Anosov.

Partially hyperbolic means $TM = \mathbb{E}^u \oplus \mathbb{E}^c \oplus \mathbb{E}^s$. In dimension 3 we could state the above for a dominated splitting.

Partial hyperbolicity result

Note: In the preprint we assume dynamical coherence (E^c is integrable).

- We show this implies there is a codimension-one foliation without Reeb components.
- This implies the manifold is irreducible.
- This then implies $M = \mathbb{T}^3$ so then Anosov.

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Dynamical coherence can be removed by recent result of Burago and Ivanov. Says that there are smooth foliations as near E^c as one wants. This then implies again no Reeb components.

Open problems

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2. If M is a torus and f is QAD on M, is f Anosov? (**Note:** This is conjectured by Mañé.)
3. If f is QAD and partially hyperbolic (dominated splitting), is f Anosov?
4. If M is 4-dimensional and f is QAD on M, what can be said about structure of M and action of f? (There are partial results by J. Rodriguez-Hertz, Ures, and Vieitez.)

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