Robustly transitive diffeomorphisms

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Dynamical systems

The setting for a dynamical system is:

- a space,
- a time, and
- a time evolution.

The space is called a **phase space** (X or M) and represents the possible states of the system. The time is either **continuous** ($\mathbb{R}$) or **discrete** ($\mathbb{Z}$). The time evolution is given by a **flow** (Φ_t) in the continuous case and a **map** (f) in the discrete case.

2 main concepts

In studying (X, f) there are 2 tools that are frequently used.

1. Look for **invariant sets**. That is $\Lambda \subset X$ such that $f(\Lambda) = \Lambda$.
2. Decompose X into invariant sets $\Lambda_1, \ldots \Lambda_k$ and **wandering** components (i.e. - points that under a forward orbit approaches one of the invariant sets and under backward iterates approaches another set).

Transitivity

Definition: A system (X, f) is **transitive** if there exists some $x \in X$ such that $\mathcal{O}^+(x)$ is dense in X (where $\mathcal{O}^+(x) = \{f^n(x) \,|\, n \in \mathbb{N}\}$).

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Example: Let $X = S^1$ and $f_\alpha(x) = (x + \alpha) \bmod 1$. This is called a circle rotation. An exercise, is that if α is irrational, then (S^1, f_α) is transitive. In fact, in this case every point has a forward orbit that is dense.

Another definition of transitivity

Proposition

Let $f : X \to X$ be continuous and X be a compact metric space. Then f is transitive if and only if for any open set U and V in X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.

Note: One direction is trivial.

To see the other let $\{V_i\}_{i=1}^{\infty}$ be a basis for the topology. Then $\bigcup_{n\in\mathbb{N}} f^{-n}(V_i)$ is dense in X for each i since it intersects every open set. So $Y = \bigcap_{i\in\mathbb{N}}(\bigcup_{n\in\mathbb{N}} f^{-n}(V_i))$ is a dense set of points. If $y \in Y$, then for each i we know $y \in f^{-n}(V_i)$ for some n and $f^n(y) \in V_i$. Then $\mathcal{O}^+(y)$ is dense in X.

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Remark

X need only be locally compact and Hasdorff for the above proposition.

Topology on the set of diffeomorphisms

Let M be a **compact, smooth, boundaryless, connected manifold**. We let $\mathrm{Diff}(M)$ be the set of diffeomorphisms from M to M. (A diffeomorphism is a continuous bijection with where $Df(x)$ an $Df^{-1}(x)$ are defined and invertible for all $x \in M$.)

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We want to define a metric on this set. For $f, g \in \mathrm{Diff}(M)$ we set

$$d_1(f, g) = \sup_{x \in M} \{d(f(x), g(x)), \|Df(x) - Dg(x)\|\}.$$

It is not hard to see that this defines a metric. So f and g are close if **points are moved uniformly close to one another** and **the derivatives stay close.**

Note: With this topology the set $\mathrm{Diff}(M)$ is an infinite dimensional metric space.

Robustly transitive diffeomorphisms

Definition: A diffeomorphism f is **robustly transitive** if there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}(M)$ such that each $g \in \mathcal{U}$ is transitive.

Remark

A circle rotation f_α is transitive if α irrational, but f_α is not robustly transitive. To see this notice that if α is rational, then every point in S^1 is periodic under f_α. (i.e. - $f_\alpha^n(x) = x$ for some $n \in \mathbb{N}$ and all $x \in S^1$.)

Comments on robust transitivity

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1. In an attempt to understand the behavior of the systems (M, f) where $f \in \mathrm{Diff}(M)$ we know that robust transitivity is an open condition. Hence, this is an **important component of the global dynamics.**
2. The dynamics of these systems are **very interesting**. There is a great deal of complexity seen in these systems that is still not well understood. In 2-dimensional systems there is a good understanding of the systems, but in higher dimensions there is still much that is unknown.
3. Although the condition is topological the theory is closely related to stable (robust) ergodicity a measurable (and probabilistic) concept.

Hyperbolic periodic points

To show robust transitivity we will see it is useful to look at hyperbolic periodic points.

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A periodic point p of period n is **hyperbolic** *if $Df^n p$ has no eigenvalues on the unit circle.*

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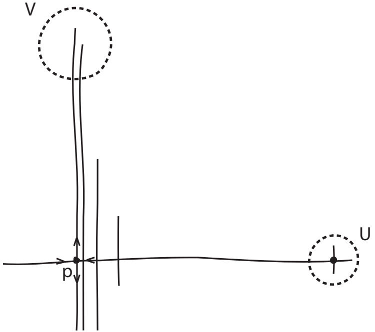
Remark:

- $T_pM = E^s \oplus E^u$ where each vector $v \in E^s$ is uniformly exponentially contracted under Df^n and each vector $v \in E^u$ is uniformly exponentially expanded under Df^n.
- $W^s(p) = \{x \in M : d(f^{in}(x), f^{in}(p)) \to 0, i \to \infty\}$ is an immersed copy of Euclidean space and tangent at p to E^s. (Similarly, for E^u using f^{-1}.)

Basic transitivity criterium

Lemma

If f has a hyperbolic periodic point p whose stable and unstable manifolds are (robustly) dense in M, then f is (robustly) transitive.



Example - hyperbolic toral automorphisms

The first known examples of robustly transitive diffeomorphisms were hyperbolic toral automorphisms.

Definition: An $n \times n$ matrix A is a **hyperbolic toral automrophism** if it has

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Remark: Any such A induces a diffeomorphism of the n-torus, denoted f_A. For f_A one can show that the stable and unstable manifolds of the origin are robustly dense in the n-torus.

2,1,1,1

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. This matrix has has determinant 1 and two eigenvalues $\lambda_1 = (3+\sqrt{5})/2 > 1$ and $\lambda_2 = 1/\lambda_1$. The eigenvectors of A are

$$v_{\lambda_1} = ((1+\sqrt{5})/2, 1) \text{ and } v_{\lambda_2} = ((1-\sqrt{5})/2, 1).$$

Since the slopes are irrational we know that the projection of subspaces spanned by eigenvectors are dense. So the origin has stable and unstable manifolds that are dense in the 2-torus.

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Remark: Using a condition called structural stability the transitivity can be shown to be robust.

Other examples

In the 1970's, Shub and Mañé constructed examples of non-hyperbolic robustly transitive diffeomorphisms.

- Although, these are not the hyperbolic toral automorphisms both of these examples were constructed as C^0 perturbations of hyperbolic toral automorphisms.
- The idea is to change the dynamics, quite drastically, in a neighborhood, but do this in such a way that the basic transitivity criterium still holds.

In the 1990's new examples were given by Bonatti and Díaz, and Bonatti and Viana. These new examples are in many respects much more complicated.

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Theorem

(Díaz and F.) There is a C^1-residual set of certain non-hyperbolic robustly transitive diffeomorphisms that have no symbolic extensions.

Shub example - part 1

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (This construction can be carried out by more general hyperbolic toral autormorphisms) and $F_0 : \mathbb{T}^4 \to \mathbb{T}^4$ be defined as $F_0(x, y) = (A^2 x, Ay)$.

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We now want to change the map. Let $F(x, y) = (A^2x, f_x(y))$ where $f_x \in \mathrm{Diff}(\mathbb{T}^2)$ and depends smoothly on x. Such a map is called a **skew product**. The space $\mathbb{T}^2 \times \{0\}$ is called the **base** and each $\{x\} \times \mathbb{T}^2$ is called a **fiber**.

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Remark: We will add conditions to f_x.

Shub example -part 2

Let p be a fixed point for A and q be a different fixed point for A^2. Let U be a very small neighborhood in $\mathbb{T}^2$ containing q. (So it will not contain p.) Let $f_x = A$ for $x \notin U$.

At q we want f_q to be a DA-diffeomorphism where the perturbation is done about (q, p). (I will describe a DA-diffeomorphism later.)

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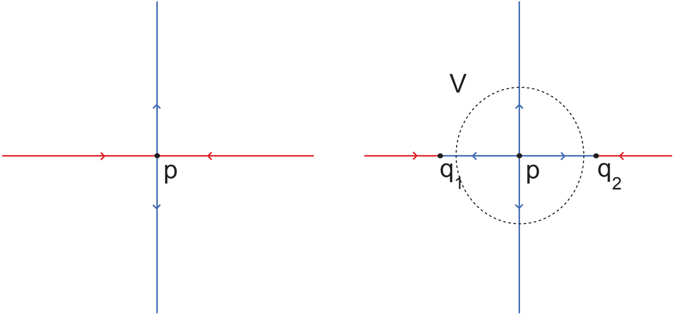
By the domination of A^2 over A this can be shown to be robust (i.e. - using what is called normal hyperbolicity).

DA -diffeomorphisms

Perturb A in neighborhood of p. This is done so the stable direction remains unchanged, but at p the stable direction becomes expanding (p becomes a source). Then 2 new fixed points are formed with 1-stable and 1-unstable direction.



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What are the mechanisms from which robust transitivity arises?