Symbolic extensions for partially hyperbolic diffeomorphisms

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Joint work with L. J. Díaz, M. J. Pacifico, and J. Vieitez

Markov partition for Anosov diffeomorphisms

Adler and Weiss constructed Markov partitions for Anosov diffeomorphisms of surfaces and then Sinai ('68) constructed Markov partitions for general Anosov diffeomorphisms. Where a Markov partition is defined as follows:

A *Markov partition* of Λ is a finite collection of proper rectangles $\{R_i\}_{i=1}^M$ covering Λ such that

1. if $i \neq j$, then $\mathrm{int}(R_j) \cap \mathrm{int}(R_i) = \emptyset$,
2. if $z \in \mathrm{int}(R_i)$ and $f(z) \in \mathrm{int}(R_j)$, then $W^u(f(z), R_j)) \supset f(W^u(z, R_i))$ and $f(W^s(z, R_i)) \subset W^s(f(z), R_j)$.

There is a natural subshift of finite type $\Sigma \subset \Sigma_M$ associated with the Markov parittion.

Weak forms of hyperbolicity

Question: What happens when we weaken the hyperbolicity? For instance if a diffeomorphism is partially hyperbolic? or nonuniformly hyperbolic?

It is not hard to construct examples of partially hyperbolic diffeomorphisms with no Markov partition. However, what if we weaken some of the requirements of a Markov partition?

Symbolic extensions

A *symbolic extension* of a dynamical system (X, f) is a subshift (Σ, σ) and a semiconjugacy $\phi : \Sigma \to X$ $(f\phi = \phi\sigma)$.

Remark:

- In this case Σ need not be a subshift of finite type. (A subshift is a closed shift invariant subset of a full shift.)
- The map ϕ need not be finite-to-one.
- A Markov partition has a natural subshift extension.
- Subshifts have nice properties such as: expansivity, uppersemicontinuity of the entropy function $\mu \to h_\mu(\Sigma)$, and finite topological entropy.

Interest in symbolic models

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Classical examples of models using symbolic dynamics are the following:

- Describing the homotopy class of a trajectory of a geodesic flow on a surface of negative curvature (Hadamard, Morse).
- Parameterizing a unimodal map on $[0, 1]$ by the kneading sequence.
- Krieger's generator theorem says that every ergodic measure-preserving invertible transformation with finite entropy has a finite generating partition $\mathcal{P}$ (measure-theoretically isomorphic to the symbolic system represented by the shift map on the $\mathcal{P}$ names).

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Remark: Existence of a symbolic extensions relates to the entropy structure as defined by Downarowicz.

Diffeomorphisms with no symbolic extensions

Theorem:(Downarowicz, Newhouse, '05) Let M be a compact orientable surface, ω be a symplectic form on M, and $\mathrm{Diff}_\omega(M)$ denote the set of diffeomorphisms preserving ω. Then there is a C^1 residual set of set of diffeomorphisms of $\mathrm{Diff}_\omega(M)$ such that each diffeomorphism is Anosov or has no symbolic extension.

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Theorem: (Asaoka, '08) For any manifold M with $\dim(M) \geq 3$ there exists an open subset $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ and a C^1 residual set $\mathcal{R} \subset \mathcal{U}$ such that each $f \in \mathcal{R}$ has no symbolic extension.

Asymptotic *h*-expansive and symbolic extensions

Boyle, D. Fiebig, and U. Fiebig were able to show that asymptotic *h*-expansivity of a system is equivalent to the existence of a "nice" symbolic extension called a principal symbolic extension. (A symbolic extension (Σ, σ) for a system (X, f) is a *principal symbolic extension* if $h_\nu(\Sigma) = h_{\phi_*\nu}(f)$ for every invariant measure ν of Σ.)

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Note: This implies the identity on a manifold has a symbolic extension.

Further systems with symbolic extensions

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Question: Suppose f is a C^r diffeomorphism of a compact Riemannian manifold, with $1 < r < \infty$. When does f have a symbolic extension?

Note:

- The examples of Downarowicz and Newhouse and Asaoka have superexponential growth of periodic points.
- However, superexponential growth of periodic points is not necessarily sufficient to preventing a symbolic extension as shown in our first theorem.

Symbolic extensions for partially hyperbolic diffeomorphisms with 1-dimensional center bundle

Theorem 1:(Díaz, F., Pacifico, Vieitez) Every partially hyperbolic diffeomorphism with a center bundle that splits into 1-dimensional subbundles is *h*-expansive (entropy expansive) and therefore has a principal symbolic extension.

Note: By partially hyperbolic we mean TM splits into one of the following forms: $E^s \oplus E^c$, $E^s \oplus E^c \oplus E^u$, or $E^c \oplus E^u$.

Topological entropy of a subset

For a set $Y \subset X$ a set $A \subset Y$ is (n, ϵ)-*spanning* if for any $y \in Y$ there exists a point $x \in A$ where $d_n(x, y) < \epsilon$ where $d_n(x, y) = \max_{0 \le i \le n}(f^i(x), f^i(y))$. The minimum cardinality of an (n, ϵ)-spanning set of Y is denoted $r_n(Y, \epsilon)$.
Given a subset $Y \subset X$ we let

$$\bar{r}(Y, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(Y, \epsilon) \quad \text{and} \quad \tilde{h}(f, Y) = \lim_{\epsilon \to 0} \bar{r}(Y, \epsilon).$$

h-expansive

We denote the closed ball with center at x and radius ϵ in the d_n metric as $B^n_\epsilon(x)$. Let

$$\Phi_\epsilon(x) = \bigcap_{n=1}^{\infty} B^n_\epsilon(x).$$

Then

$$h^*_f(\epsilon) = \sup_{x\in X} \tilde{h}(f, \Phi_\epsilon(x)).$$

The map f is *h-expansive* if

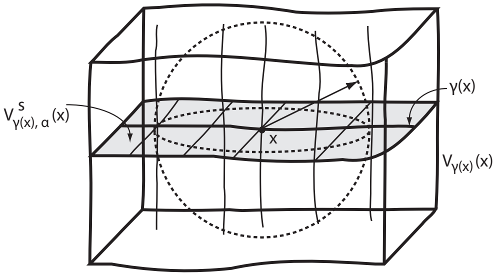
$$h^*_f(\epsilon) = 0$$

for all $\epsilon > 0$ sufficiently small. The map f is *asymptotically h-expansive* if

$$\lim_{\epsilon\longrightarrow 0} h^*_f(\epsilon) = 0.$$

Outline of the proof of Theorem 1

We show there is a uniform scale such that for every center curve (not necessarily unique), $\gamma(x)$, there is a foliation chart, $V_{\gamma(x)}(x)$ at each point (see figure).



Outline of the proof of Theorem 1 -part 2

Lemma

For every $x \in M$ and ϵ small enough it holds that

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Remark: So $\Phi_\epsilon(x) \subset W^{sc}_\epsilon(x) \subset V^s_{\gamma(x),\alpha}(x)$. Show that there is no exponential growth in stable direction of spanning set. So growth occurs in 1-dimensional center of uniformly bounded length. So entropy is zero on small scale.

Condition for no symbolic extension

Let $\mathcal{M}(f)$ be the set of f-invariant Borel probability measures for f. Fix a sequence of partitions $\{\alpha_k\}$ (essential sequence: i.e. diameters go to zero and boundaries have measure zero) and denote $h_k(\nu) = h_\nu(\alpha_k)$ for $\nu \in \mathcal{M}(f)$.

Proposition: (Downarowicz, Newhouse, '05) Suppose that $\mathcal{E}$ is a compact subset of $\mathcal{M}(f)$ such that there is a positive real number ρ_0 such that for each $\mu \in \mathcal{E}$ and each $k > 0$,

$$\limsup_{\nu \in \mathcal{E}, \nu \to \mu} [h_\nu(f) - h_k(\nu)] > \rho_0.$$

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Then f has no symbolic extension.

Note: So on arbitrarily small scale we see entropy created, but not with respect to the partition.

No symbolic extension for certain robustly transitive sets

Let U be an open set in a closed manifold M. Let $\mathcal{T}(U)$ be the set of diffeomorphisms, f, such that $\Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U})$ is robustly transitive.

Let $\mathcal{T}^{nh}(U)$ denote the set of diffeomorphisms, f, such that $\Lambda_f(U)$ is not hyperbolic. We let $\mathcal{T}_2^{nh}(U)$ be the subset of $\mathcal{T}^{nh}(U)$ such that for each $f \in \mathcal{T}_2^{nh}(U)$ there is a neighborhood $\mathcal{V}$ of f such that for each $g \in \mathcal{V}$ the set $\Lambda_g(U)$ has a non-hyperbolic center indecomposable bundle of dimension at least 2.

Theorem 2:(Díaz, F. , Pacifico, Vieitez) There is a C^1 residual set $\mathcal{R}$ in $\mathcal{T}_2^{nh}(U)$ such that each diffeomorphism in $\mathcal{R}$ has no symbolic extension.

Outline of the proof of Theorem 2

- Explain that the existence of center indecomposable bundle of dimension at least 2 generates persisitent homoclinic tangencies.

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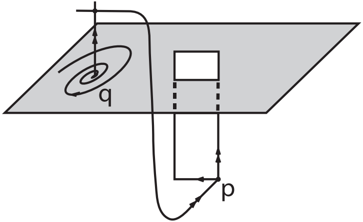
- Look at saddles with real multipliers and show tangencies occur in a local normally hyperbolic surface.
- Show how nonexistence follows by building up horseshoes in a small neighborhood with "high" entropy. These horseshoes are contained in elements of the α_n partition and have entropy near a positive constant.

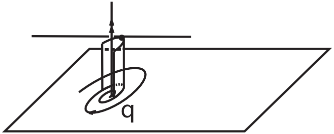
Existence of tangencies

Proposition

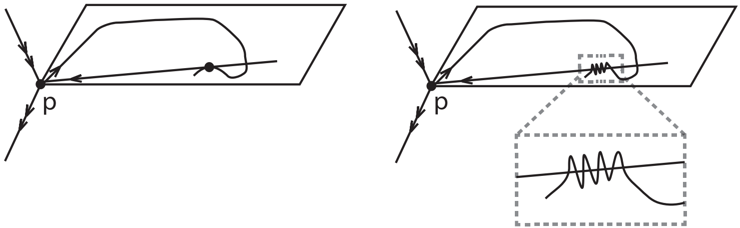
Let $\Lambda_f(U)$ be a robustly transitive set having an indecomposable center bundle of dimension at least 2.
Then there is a saddle $p_f \in \Lambda_f(U)$, a neighborhood of f, $\mathcal{U}_f$, and a dense subset $\mathcal{T}$ of $\mathcal{U}_f$ consisting of diffeomorphisms with homoclinic tangencies associated to p_g.

Outline of the proof of Theorem 2 - figure1





Outline of the proof of Theorem 2 - figure 2



h-expansive homoclinic classes

Let p be a hyperbolic periodic point for $f \in \mathrm{Diff}(M)$. Then the *homoclinic class* of p is $H(p) = \overline{W^u(\mathcal{O}(p)) \pitchfork W^s(\mathcal{O}(p))}$.

Theorem 3:(Díaz, F., Pacifico, Vieitez) Let $f \in \mathrm{Diff}(M)$ and p be a hyperbolic periodic point for f such that there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}(M)$ where $H(p_g)$ (the continuation of $H(p)$) is h-expansive for all $g \in \mathcal{U}$. Then $H(p_g)$ has a dominated splitting, $T_{H(p_g)}M = E \oplus F_1 \oplus \cdots \oplus F_k \oplus G$ where all F_j are one dimensional and nonhyperbolic.

Note: For a diffeomorphism f of M a compact f-invariant set Λ has a *dominated splitting* if $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$ where each E_i is non-trivial and Df-invariant for $1 \leq i \leq k$ and there exists an $m \in \mathbb{N}$ such that $\|Df^n|_{E_i(x)}\| \, \|(Df^n|_{E_j(x)})^{-1}\| \leq \frac{1}{2}$ for every $n \geq m$, $i > j$, and $x \in \Lambda$.

Idea of proof of Theorem 3

- Assume no dominated splitting. Then there is a flat tangency. So create horseshoes as in Theorem 2 and show not h-expansive, a contradiction.
- If dominated splitting has nonhyperbolic 2-dimensional subspace, then again using arguments as in Theorem 2 there is a contradiction.

Theorem 4:(Díaz, F., Pacifico, Vieitez) Let $f \in \mathrm{Diff}(M)$ and p be a hyperbolic periodic point for f such that there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}(M)$ where $H(p_g)$ (the continuation of $H(p)$) is h-expansive for all $g \in \mathcal{U}$ and $H(p_g)$ is isolated, then $T_{H(p_g)}M = E \oplus F_1 \oplus \cdots \oplus F_k \oplus G$ where all F_j are one dimensional and nonhyperbolic and both E and G are nonempty and hyperbolic for all $g \in \mathcal{U}$.

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Idea of proof: If $H(p)$ is isolated, then use generic arguments to see indices of the periodic points give the hyperbolic subbundles E and G similarly to the arguments mentioned previously.