Math 313
Section 9H

MIDTERM 2

Name: ____________________________

February 25,26,28 2011

Answer all questions and show all your work carefully. Graphic Calculators are not allowed, but a regular scientific one will be permitted. There is a time limit of three hours for this test.

I have been quoted as saying, "do the best you can". But I want to emphasize that it be the very best. We are too prone to be satisfied with mediocre performance. We are capable of doing so much better.
Pres. Gordon B. Hinckley

Prof. Vianey Villamizar

Problem No.	**Points**
1.-)	
2.-)	
3.-)	
4.-)	
5.-)	
Total	

1. **a) (8 points)** Compute $det(A)$, where

$$A = \begin{pmatrix} 1 & -5 & 2 & 7 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 3 & 5 \\ 3 & 15 & 7 & -2 \end{pmatrix}$$

b) 12 points) Show that a square matrix A is invertible if and only if $det(A) \neq 0$.
Hint: If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then $det(EB) = det(E)det(B)$.

a)

$$A \sim \begin{pmatrix} 1 & -5 & 2 & 7 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & 30 & 1 & -23 \end{pmatrix}, \text{ then}$$

$$det(A) = 1 \begin{vmatrix} 1 & -3 & 6 \\ 0 & 3 & 5 \\ 30 & 1 & -23 \end{vmatrix} = 1 \begin{vmatrix} 1 & -3 & 6 \\ 0 & 3 & 5 \\ 0 & 91 & -203 \end{vmatrix} = 1 \begin{vmatrix} 3 & 5 \\ 91 & -203 \end{vmatrix} =$$

$$= -609 - 455 = \boxed{-1064}$$

b) ($\rightarrow$) If A is invertible, then the row reduced echelon form $R = I$. It means

$$E_r \dots E_2 E_1 A = R = I,$$

Then

$$1 = det(I) = det(E_r \dots E_1 A) = det(E_r(E_{r-1} \dots E_1 A)) =$$

$$= det(E_r)\, det(E_{r-1} \dots E_1 A) = \dots =$$

$$= det(E_r)\, det(E_{r-1}(E_{r-2} \dots E_1 A)) = \underset{\text{r-3 steps}}{\overset{\text{after}}{\dots}} =$$

$$= det(E_r)\, det(E_{r-1})\, det(E_{r-2} \dots E_1 A) = \dots =$$

$$= det(E_r)\, det(E_{r-1}) \dots det(E \dots det(E_1 A) =$$

$$= det(E_r)\, det(E_{r-1}) \dots det(E_2)\, det(E_1)\, det(A)$$

2. a) (7 points) Find an equation of the plane with normal vector $\mathbf{n} = (1, 2, 3)$ and passing through the point $P_0(-1, 2, 5)$.

b) (8 points) Is the point $(-1, 4, -2)$ in the plane obtained in (a)?. Why? If not, find an equation for the line passing through $(-1, 4, -2)$ and perpendicular to the plane in (a).

c) (5 points) What is the distance from point $(-1, 4, -2)$ to the plane obtained in (a)?

a) $\vec{n} = (1, 2, 3)$, $P_0(-1, 2, 5)$

$$\vec{n} \cdot (x+1, y-2, z-5) = 0$$

$$\Leftrightarrow \quad x + 1 + 2(y-2) + 3(z-5) = 0$$

$$\text{or} \quad \boxed{x + 2y + 3z = 18.} \qquad (1)$$

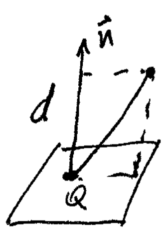
b) Subst. leads to $-1 + 8 - 6 = 1 \neq 18 \Rightarrow (-1, 4, -2)$ is not in plane.

Line through $(-1, 4, -2)$ perpendicular to plane (1)

$$\vec{x} = t\,\vec{n} + (-1, 4, -2)$$

$$\text{or} \quad (x, y, z) = t\,(1, 2, 3) + (-1, 4, -2) \Leftrightarrow \begin{cases} x = t - 1 \\ y = 2t + 4 \\ z = 3t - 2 \end{cases}$$

c) Distance from point $(-1, 4, -2)$ to plane (1).



Choose $Q(x, y, z)$ in plane (1)

then $d = \| \mathrm{proj}_{\vec{n}} \vec{QP_1} \| = \left\| \left(\vec{QP_1} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right) \frac{\vec{n}}{\|\vec{n}\|} \right\|$

$$= \left| \vec{QP_1} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right| \underbrace{\left\| \frac{\vec{n}}{\|\vec{n}\|} \right\|}_{=1} = \left| \vec{QP_1} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right| = \frac{1}{\|\vec{n}\|} \left| \vec{QP_1} \cdot \vec{n} \right|$$

$$= \frac{|x + 1 + 2(y-4) + 3(z+2)|}{\sqrt{1^2 + 2^2 + 3^2}}, \quad \text{if } x = 0, y = 0 \Rightarrow z = 6$$

$$\text{and } d = \frac{1 - 8 + 24}{\sqrt{14}} = \frac{17}{\sqrt{14}} = \frac{17\sqrt{14}}{\sqrt{14}}$$

3. Indicate whether the statement is always true or sometimes false. Justify your answer with **a detailed general proof** or a counterexample.

a) (4 points) If A is a square matrix, then $det(A) = det(A^T)$.

b) (4 points) If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

c) (4 points) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then $||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2)$.

d) (4 points) If A is a $n \times n$ matrix and $det(A) = 5$, then $det(2A) = 10$.

e) (4 points) The scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is a vector that is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

a) True

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Use induction (best case)
or say since transposing a matrix change rows to columns and viceversa
Cofactor expansion along any row of A same as cofactor expansion along corresponding column of A^T.

b) False

$\vec{u} = (1,1)$, $\vec{v} = (1,0)$, $\vec{w} = (0,1)$

$\Rightarrow \vec{u}\cdot\vec{v} = 1 = \vec{u}\cdot\vec{w}$, but $\vec{v} \neq \vec{w}$.

c) True, $||\vec{u}+\vec{v}||^2 + ||\vec{u}-\vec{v}||^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) =$

$= \vec{u}\cdot\vec{u} + \cancel{2\vec{u}\cdot\vec{v}} + \vec{v}\cdot\vec{v} + \vec{u}\cdot\vec{u} - \cancel{2\vec{u}\cdot\vec{v}} + \vec{v}\cdot\vec{v} =$

$= 2\,\vec{u}\cdot\vec{u} + 2\,\vec{v}\cdot\vec{v} = 2\,||u||^2 + 2\,||\vec{v}||^2 = 2\,(||\vec{u}||^2 + ||\vec{v}||^2)$

d) False, $A = \begin{bmatrix} 5 & 5 \\ 1 & 2 \end{bmatrix} \Rightarrow det(A) = 5$, but $det(2A) = \begin{vmatrix} 10 & 10 \\ 2 & 4 \end{vmatrix} = 20$

e) False, $\vec{u}\cdot(\vec{v}\times\vec{w})$ is not a vector.

4.- a) **(6 points)** For **u v** and **w** in R^3, prove the following identity
$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -(\mathbf{u}\times\mathbf{w})\cdot\mathbf{v}$

b) **(4 points)** For **u** and **v** in R^3, show that if $\mathbf{u}\cdot\mathbf{v}\neq 0$, then $\tan\theta = \frac{\|\mathbf{u}\times\mathbf{v}\|}{\mathbf{u}\cdot\mathbf{v}}$, where θ is the angle between **u** and **v**.

c) **(10 points)** Let **u**=(6,4,-8), and **a**=(3,5,1). Find the vector component of **u** along **a**, and the vector component of **u** orthogonal to **a**.

$$\vec{u}\cdot(\vec{v}\times\vec{w}) =$$

a) $$(u_1,u_2,u_3)\cdot(v_2w_3-v_3w_2,\ v_3w_1-v_1w_3,\ v_1w_2-v_2w_1)$$

$$= u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1$$

Also

$$-(\vec{u}\times\vec{w})\cdot\vec{v} =$$

$$-(u_2w_3-u_3w_2,\ u_3w_1-u_1w_3,\ u_1w_2-u_2w_1)\cdot(v_1,v_2,v_3) =$$

$$= -(u_2w_3v_1 - u_3w_2v_1 + u_3w_1v_2 - u_1w_3v_2 + u_1w_2v_3 - u_2w_1v_3)$$

Same!

b) $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$, $\quad \|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\|\vec{u}\times\vec{v}\|}{\vec{u}\cdot\vec{v}\neq 0} \Leftrightarrow \tan\theta = \frac{\|\vec{u}\times\vec{v}\|}{\vec{u}\cdot\vec{v}} \checkmark$$

c) $\vec{u} = \vec{w}_1 + \vec{w}_2$, where $\underline{\vec{w}_1 = proj_{\vec{a}}\vec{u}}$ and $\underline{\vec{w}_2 = u - \vec{w}_1}$.

$$\vec{w}_1 = \left(\vec{u}\cdot\frac{\vec{a}}{\|\vec{a}\|}\right)\frac{\vec{a}}{\|\vec{a}\|} = \left((6,4,-8)\cdot\frac{(3,5,1)}{\sqrt{35}}\right)\frac{(3,5,1)}{\sqrt{35}} =$$

$$= \left(\frac{18+20-8}{35}\right)(3,5,1) = \frac{30}{35}(3,5,1) \Rightarrow \boxed{w_1 = \frac{6}{7}(3,5,1)}$$

and $$\vec{w}_2 = (6,4,-8) - \frac{6}{7}(3,5,1) = \left(\frac{24}{7}, \frac{-2}{7}, \frac{-62}{7}\right)$$

5.- For parts (a) and (b) of this question use the ten vector space axioms attached to this test.

a) (10 points) For V a vector space, $\mathbf{u}$ a vector in V, and k a scalar, show that if $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = 0$.

b) (10 points) Determine if the set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y'), \qquad k(1, y) = (1, ky),$$

is a vector space. If this set is not a vector space identify the axioms that fail.

a) Assume $k\vec{u} = \vec{0}$, then there are two possibilities for k:

i) $k = 0$, and the theorem is proven.

or ii) $k \neq 0$, then $\frac{1}{k}$ is a scalar also.

multiplying both sides of $k\vec{u} = \vec{0}$ by $\frac{1}{k}$

$$\underbrace{\frac{1}{k}(k\vec{u})}_{A} = \underbrace{\frac{1}{k}\vec{0}}_{B}$$

Now, $A \overset{9}{=} \left(\frac{1}{k}k\right)\vec{u} = 1\vec{u} \overset{10}{=} \vec{u}$

also, $B = \frac{1}{k}\vec{0} \overset{\text{previous thm}}{=} \vec{0}$ then $A = B \Rightarrow \vec{u} = \vec{0}$ ✓

b) This is a vector space. Show that all axioms are verified.

5.1 REAL VECTOR SPACES

In this section we shall extend the concept of a vector by extracting the most important properties of familiar vectors and turning them into axioms. Thus, when a set of objects satisfies these axioms, they will automatically have the most important properties of familiar vectors, thereby making it reasonable to regar these objects as new kinds of vectors.

Vector Space Axioms The following definition consists of ten axioms. A you read each axiom, keep in mind that you have already seen each of them as part of various definitions and theorems in the preceding two chapters (for instance, se Theorem 4.1.1). Remember too, you do not prove axioms; they are simply the "rules c the game."

Definition

Let V be an arbitrary nonempty set of objects on which two operations are defined, addition and multiplication by scalars (numbers). By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scala k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and l, then we call V a ***vector space*** and we call the objects in V ***vectors***.

(1) If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.

(2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

(4) There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=$ O for all $\mathbf{u}$ in V.

(5) For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such th $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.

(6) If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.

(7) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

(8) $(k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$

(9) $k(l\mathbf{u})=(kl)(\mathbf{u})$

(10) $1\mathbf{u}=\mathbf{u}$