

Math 313
Section 9

MIDTERM 3

Name: _____

Apr 4-6 2011

Answer all questions and show all your work carefully. Graphic Calculators are not allowed, but a regular scientific one will be permitted. There is a time limit of three hours for this test.

“Here we are doing what is not done in any other major university of which I am aware. We are demonstrating that faith in the Almighty can accompany and enrich scholarship in the secular. It is more than an experiment. It is an accomplishment.”

Pres. Hinckley at the Inauguration of Pres. Samuelson

Prof. Vianey Villamizar

Problem No.	Points
1.-)	
2.-)	
3.-)	
4.-)	
5.-)	
Total	

1. Indicate whether the statement is always true or sometimes false. Justify your answer with a **detailed general proof** or a counterexample.
- a) (4 points) Any three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2$ and \mathbf{v}_3 in \mathbb{R}^3 form a basis for \mathbb{R}^3 .
- b) (4 points) If A is not a square matrix then the row vectors of A must be linearly dependent.
- c) (4 points) If the size of a matrix A is 5×9 and $\text{rank}(A)=2$, then the nullity of A^T is 3.
- d) (4 points) Let \mathbf{u}, \mathbf{v} be vectors in the inner product space V , then $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal.
- e) (4 points) If A is an invertible matrix then $\lambda = 0$ is not an eigenvalue.

a) True, (1)

\mathbb{R}^3 has dim 3. Therefore, any set of 3 linearly indep. vectors (3) form a basis. Otherwise, the set of lin. indep. vector can be extended to four vectors and $\dim(\mathbb{R}^3) \geq 4$ Contradiction.

b) False, (1)

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} \vec{r}_1 \\ \vec{r}_2 \end{matrix}$$

Clearly \vec{r}_1 and \vec{r}_2 are lin. indep. because they are not multiples of each other. (3)

c) True, (1) Since $[A^T]_{9 \times 5} \Rightarrow 5 = \text{rank}(A^T) + \text{nullity}(A^T)$

(3) but $\text{rank}(A^T) = \dim(\text{Col}(A^T)) = \dim(\text{row}(A)) = \text{rank}(A) = 2$
 $\Rightarrow \text{nullity}(A^T) = 5 - 2 = 3.$

d) False (1) $\langle \vec{u} + \vec{v}, \vec{u} - \vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle - \langle \vec{v}, \vec{v} \rangle \neq 0$.
 in general. (3)

e) True, (1) If $\lambda = 0$ were an eigenvalue. It would exist $\vec{x} \neq 0$ such that $A\vec{x} = \vec{0} \Rightarrow A$ would not be invertible (equivalence thm). (3)

2. Give a detailed and complete proof of the following short theorems

a) (6 points) Let $B = \{v_1, \dots, v_n\}$ form an orthonormal basis for an inner product space V and $u \in V$, then

$$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \dots + \langle u, v_n \rangle v_n$$

b) (8 points) For any matrix A ($m \times n$), the linear system $Ax = b$ is consistent if and only if the coefficient matrix A and the augmented matrix $[A|b]$ have the same rank.

c) (6 points) Let A be an $n \times n$ matrix. Show that if $\det(A - \lambda I) = 0$, then λ is an eigenvalue of A .

d) (5 points) If $B = \{v_1, \dots, v_n\}$ is an orthogonal set of nonzero vectors in an inner product space V , then B is linearly independent.

a) Since B is a basis

$$\vec{u} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n$$

$$\Rightarrow \langle \vec{u}, \vec{v}_1 \rangle = \langle \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n, \vec{v}_1 \rangle = \alpha_1 \langle \vec{v}_1, \vec{v}_1 \rangle + \alpha_2 \langle \vec{v}_2, \vec{v}_1 \rangle + \dots + \alpha_n \langle \vec{v}_n, \vec{v}_1 \rangle$$

$\overset{=1}{\langle \vec{v}_1, \vec{v}_1 \rangle} + \overset{=0}{\langle \vec{v}_2, \vec{v}_1 \rangle} + \dots + \overset{=0}{\langle \vec{v}_n, \vec{v}_1 \rangle}$

$$= \alpha_1$$

Similarly, $\langle \vec{u}, \vec{v}_2 \rangle = \alpha_2, \dots, \langle \vec{u}, \vec{v}_n \rangle = \alpha_n.$

Then, $\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \dots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n.$

b) $A = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \end{bmatrix}_{m \times n}$ and $[A|\vec{b}] = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n & \vec{b} \end{bmatrix}_{m \times n+1}$

(\leftarrow) If $\text{rank}(A) = k \Rightarrow \dim(\text{Col}(A)) = k$
 If $\text{rank}([A|\vec{b}]) = k \Rightarrow \dim(\text{Col}([A|\vec{b}])) = k$

Thus, $\vec{b} = \beta_1 \vec{c}_1 + \beta_2 \vec{c}_2 + \dots + \beta_n \vec{c}_n$
 $\Rightarrow A \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \vec{b}$, which implies that the system $A\vec{x} = \vec{b}$ is consistent

$\Rightarrow \vec{b}$ can be written as a linear comb. of vectors in $\text{Col}(A)$, otherwise \vec{b} could be added to a basis of $\text{Col}(A)$ B , and the new set $B \cup \{\vec{b}\}$ would be a basis for $\text{Col}([A|\vec{b}])$
 $\Rightarrow \dim(\text{Col}([A|\vec{b}])) = k+1 \neq k = \dim(\text{Col}(A)).$
 Contradicting hypothesis

c) If $\det(A - \lambda I) = 0 \Rightarrow A - \lambda I$ is Singular

\Rightarrow there is $\vec{x} \neq \vec{0}$ such that $(A - \lambda I)\vec{x} = \vec{0}$

$\Rightarrow A\vec{x} = \lambda\vec{x} \Rightarrow \lambda$ is an eigenvalue of A
and \vec{x} is a corresponding eigenvector.

d) If $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n = \vec{0}$

$\Rightarrow \langle \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n, \vec{v}_1 \rangle = \langle \vec{0}, \vec{v}_1 \rangle = 0$

$\Rightarrow 0 = \langle \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n, \vec{v}_1 \rangle = \alpha_1 \langle \vec{v}_1, \vec{v}_1 \rangle + \alpha_2 \langle \vec{v}_2, \vec{v}_1 \rangle + \dots + \alpha_n \langle \vec{v}_n, \vec{v}_1 \rangle =$
 $= \alpha_1$

Similarly, $\alpha_2 = 0, \dots, \alpha_n = 0.$

b) (\Rightarrow) and If $A = [\vec{c}_1 \vec{c}_2 \dots \vec{c}_n]_{m \times n}$ is consistent then there exists

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ such that

$$x_1 \vec{c}_1 + x_2 \vec{c}_2 + \dots + x_n \vec{c}_n = \vec{b}$$

$\Rightarrow \vec{b} \in \text{col}(A) \Rightarrow \text{col}(A) = \text{span}\{\vec{c}_1, \vec{c}_2, \dots, \vec{c}_n\} =$
 $= \text{span}\{\vec{c}_1, \dots, \vec{c}_n, \vec{b}\} = \text{col}([A|\vec{b}])$

and $\text{rank}(A) = \dim(\text{col}(A)) = \dim(\text{col}([A|\vec{b}])) = \text{rank}([A|\vec{b}]).$

3. Consider the linear system $Ax = b$ where

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \\ 3 & -2 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} -6 \\ 2 \\ -2 \end{pmatrix}$$

- a) (5 points) Show that the above system is inconsistent.
 b) (8 points) Find a least squares solution of the above linear system. Is it unique? (Explain)
 c) (3 points) Find the orthogonal projection of b on the column space of A .

$$\begin{aligned} \text{a)} \quad & \begin{pmatrix} 2 & -1 & -6 \\ 2 & 4 & 2 \\ 3 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & 2 \\ 2 & -1 & -6 \\ 3 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & -6 \\ 3 & -2 & -2 \end{pmatrix} \\ & \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -8 \\ 0 & -8 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 8/5 \\ 0 & -8 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 8/5 \\ 0 & 0 & 39/5 \end{pmatrix} \quad \begin{matrix} 0 = 39/5 \\ \text{inconsistent!} \end{matrix} \end{aligned}$$

b) We need to solve the associated normal system

$$A^T A \bar{x} = A^T \bar{b} \quad A^T A = \begin{bmatrix} 2 & 2 & 3 \\ -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 0 & 21 \end{bmatrix}$$

$$A^T \bar{b} = \begin{bmatrix} 2 & 2 & 3 \\ -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} -6 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -14 \\ 18 \end{bmatrix}$$

thenⁿ least squares solution is any soln. of

$$\begin{bmatrix} 17 & 0 \\ 0 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -14 \\ 18 \end{bmatrix} \Rightarrow \boxed{x_1 = \frac{-14}{17}}, \quad \boxed{x_2 = \frac{18}{21} = \frac{6}{7}}$$

$$\text{c)} \quad \text{proj}_{\text{col}(A)} \bar{b} = \begin{pmatrix} 2 & -1 \\ 2 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -14/17 \\ 6/7 \end{pmatrix} = \begin{pmatrix} -28/17 - 6/7 \\ -28/17 + 24/7 \\ -42/17 - 12/7 \end{pmatrix} = \begin{pmatrix} -298/119 \\ 212/119 \\ -498/119 \end{pmatrix}$$

4. a) (5 points) Consider the vectors $\mathbf{u}_1 = (1, -1, 0)$, $\mathbf{u}_2 = (2, 0, 1)$, $\mathbf{u}_3 = (0, -1, 1)$. Show that the set $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ form a basis for \mathbb{R}^3 .
- b) (12 points) Show that the set S_1 is not an orthonormal basis for \mathbb{R}^3 with respect to the Euclidean inner product. Use the Gram-Schmidt process to transform the basis S_1 into an orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and find the coordinates of $\mathbf{u} = (3, -2, 4)$ with respect to S .

a) Consider

$$\alpha_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For linear independence,

We want to show that $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Equivalent to show that the linear system

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ only admits the trivial sol.}$$

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}$$

$$\Rightarrow -3\alpha_3 = 0 \Rightarrow \alpha_3 = 0 \Rightarrow \alpha_2 = -\alpha_3 = 0, \quad \alpha_1 = -2\alpha_2 = 0. \checkmark$$

Since $\dim(\mathbb{R}^3) = 3$ and the set $S_1 = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly indep. then S_1 is a basis for \mathbb{R}^3 .

b) The set S_1 is obviously not orthonormal because

$$\|\vec{u}_i\| \neq 1, \quad i=1,2,3 \text{ and also } \vec{u}_1 \cdot \vec{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 2 \neq 0$$

In the next page, we apply the Gram-Schmidt process to the set of vectors S_1 .

G-S

$$\vec{v}_1' = \vec{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{v}_2' = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1' \rangle}{\|\vec{v}_1'\|^2} \vec{v}_1' = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_3' = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1' \rangle}{\|\vec{v}_1'\|^2} \vec{v}_1' - \frac{\langle \vec{u}_3, \vec{v}_2' \rangle}{\|\vec{v}_2'\|^2} \vec{v}_2' = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - 0 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

Then $\|\vec{v}_1'\| = \sqrt{2}$, $\|\vec{v}_2'\| = \sqrt{3}$, $\|\vec{v}_3'\| = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}$

\Rightarrow the corresponding orthonormal vectors are:

$$\vec{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

Coords. of $\vec{u} = (3, -2, 4)$ with respect to $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \langle \vec{u}, \vec{v}_3 \rangle \vec{v}_3 \quad (*)$$

$$\langle \vec{u}, \vec{v}_1 \rangle = \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{5}{\sqrt{2}}, \quad \langle \vec{u}, \vec{v}_2 \rangle = \frac{3}{\sqrt{3}} - \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} = \frac{5}{\sqrt{3}}$$

$$\langle \vec{u}, \vec{v}_3 \rangle = -\frac{3}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{8}{\sqrt{6}} = \frac{7}{\sqrt{6}}$$

Therefore, $[\vec{u}]_S = \begin{pmatrix} 5/\sqrt{2} \\ 5/\sqrt{3} \\ 7/\sqrt{6} \end{pmatrix}$

5. Consider the matrix $A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 8 & 0 & 6 \end{pmatrix}$. Answer the following questions:

- a) (4 points) Find the eigenvalues of A .
- b) (10 points) Find the corresponding eigenvectors. Is it possible to obtain a set of eigenvectors $S = \{v_1, v_2, v_3, v_4\}$ linearly independent? If you say yes, find them and show that they are linearly independent.
- c) (3 points) Is the matrix A diagonalizable, $P^{-1}AP = D$? Why?
- d) (5 points) If the matrix A is diagonalizable, find P that diagonalizes A . Also find P^{-1} and D . Verify that $P^{-1}AP = D$.

a) The eigenvalues of A are $\lambda_1 = 4$, $\lambda_2 = 6$ both with multiplicity 2, because this is a lower triangular matrix.

b) Corresponding eigenvectors:

$$\boxed{\lambda_1 = 4} \quad (A - 4I)\vec{x} = 0 \quad \text{equivalent to}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 8 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_3 = -2x_2 \Rightarrow \boxed{x_3 = -x_2}$$

$$8x_2 = -2x_4 \Rightarrow \boxed{x_4 = -4x_2}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ -x_2 \\ -4x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ -4 \end{pmatrix}$$

The vectors $\vec{p}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{p}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -4 \end{pmatrix}$

are clearly lin. indep. and they form a basis for the eigenspace associated to $\lambda_1 = 4$

(c) The matrix A is diagonalizable because it has a set of 4 linearly independent eigenvectors.

Similarly for $\lambda_2 = 6$ the corresponding eigenvectors satisfy $(A - 6I)\vec{x} = \vec{0}$, which is equivalent to

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

x_3, x_4 free variables.

Then $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

The vectors $\vec{p}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{p}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis

for the eigenspace $\lambda_2 = 6$

Also, the set of eigenvectors $S = \{\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4\}$ is a linearly indep set, because \vec{p}_1 and \vec{p}_2 are lin. indep ($\lambda_1 = 4$)
 \vec{p}_3 and \vec{p}_4 are lin. indep ($\lambda_2 = 6$)

And eigenvectors corresponding to diff. eigenvalues are linearly indep. According to thm studied in class A is diagonalizable and

$$P^{-1}AP = D$$

where $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$

Verification

First, we need to find P^{-1}

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 & 1 \end{array} \right)$$

Then

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{pmatrix}$$

Then

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 8 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 6 & 6 & 0 \\ 0 & 24 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \checkmark$$