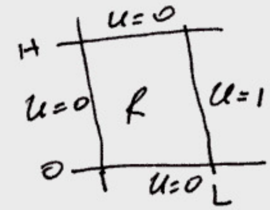
8.6.1 (b)

$$\begin{cases} \nabla^2 u = Q(x,y) \\ u \text{ on bdry} \longrightarrow \end{cases}$$



Assuming

$$u(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \, \phi_{nm}(x,y)$$

?

$$\phi_{nm}(x,y) = \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{H} y\right)$$

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2.$$

<u>Green's formula:</u>

$$\iint_R \left[u \nabla^2 \phi_{nm} - \phi_{nm} \nabla^2 u\right] dx\,dy = \int_{\partial R} \left[u \nabla \phi_{nm} \cdot \hat{n} - \overset{0}{\phi_{nm}} \nabla u \cdot \hat{n}\right] dl$$

$$\Leftrightarrow \iint_R \left[u(-\lambda_{nm} \phi_{nm}) - \phi_{nm} Q(x,y)\right] dx\,dy = \int_0^H \nabla \phi_{nm} \cdot \overset{\hat{i}}{\hat{n}} \, dy$$

$$\nabla \phi_{nm} \to \langle (\phi_{nm})_x, (\phi_{nm})_y \rangle$$

$$\Rightarrow -\lambda_{nm} \iint_R u \, \phi_{nm} \, dx\,dy = \iint_R Q(x,y) \phi_{nm} \, dx\,dy + \int_0^H \frac{\partial \phi_{nm}}{\partial x} dy \qquad (1.1)$$

8.6.1 b) (Cont.)

$$-\lambda_{nm} \iint_R \left(\sum_l \sum_q b_{lq} \phi_{lq} \right) \phi_{nm}\, dx\, dy =$$

$$= \underbrace{\iint_R Q(x,y)\, \phi_{nm}\, dx\, dy + \int_0^H \frac{\partial \phi_{nm}}{\partial x}\, dy}_{\text{rhs}}$$

orthog. $l=n,\ q=m$

$$\Rightarrow -\lambda_{nm}\, b_{nm} \iint_R \phi_{nm}^2\, dx\, dy = \text{rhs}$$

$$\Rightarrow b_{nm} = -\frac{1}{\lambda_{nm}} \left[q_{nm} + \frac{\frac{n\pi}{L} \int_0^H \left[\cos \frac{n\pi}{L} x \Big|^{=L} \sin \frac{m\pi}{H} y \right] dy}{LH/4} \right]$$

$$\frac{\frac{n\pi}{L}\left[(-1)^n \left(-\cos \frac{m\pi}{H} y \right) \frac{H}{m\pi} \right|_0^H}{LH/4}$$

$$\Rightarrow b_{nm} = -\frac{1}{\lambda_{nm}} \left[q_{nm} + \frac{n\cancel{\pi}}{L} \frac{H}{m\cancel{\pi}} \left((-1)^n \left((-1)^{m+1} + 1 \right) \right) \Big/ LH/4 \right.$$

$$(-1)^n \left(1 - (-1)^m \right)$$

$$b_{nm} = -\frac{1}{\lambda_{nm}} \left[q_{nm} + \frac{n\cancel{H}}{Lm} \frac{4}{L\cancel{H}} (-1)^n \left[1 - (-1)^m \right] \right] = \frac{-1}{\lambda_{nm}} \left[q_{nm} + \frac{4n}{mL^2} (-1)^n \left[1 - (-1)^m \right] \right]$$

$$\rightarrow q_{nm} \equiv \frac{\iint_R Q(x,y)\, \phi_{nm}\, dx\, dy}{\iint_R \phi_{nm}^2\, dx\, dy}, \quad \boxed{\iint_R \phi_{nm}^2\, dx\, dy = \int_0^L \int_0^H \sin \frac{n\pi}{L} x\, \sin \frac{m\pi}{H} y\, dx\, dy = \frac{LH}{4}}$$

If homog. BC's.

From (1.1)

$$\iint_R u\,\phi_{nm}\,dxdy = -\frac{1}{\lambda_{nm}} \iint_R Q\,\phi_{nm}\,dxdy$$

Now, $\quad u = \sum_{l,q} b_{l,q}\,\phi_{l,q}$

$\overset{\text{orthog}}{\Rightarrow} \quad b_{nm} \iint_R \phi_{nm}^2\,dxdy = -\frac{1}{\lambda_{nm}} \iint_R Q\,\phi_{nm}\,dxdy$

$$\Rightarrow \quad b_{nm} = -\frac{1}{\lambda_{nm}} \frac{\iint_R Q\,\phi_{nm}\,dxdy}{\iint_R \phi_{nm}^2\,dxdy} = -\frac{1}{\lambda_{nm}} q_{nm}$$

Therefore, Soln. of original nonhomog. BVP:

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{q_{nm} + \frac{4n}{mL^2}(-1)^n\left[1-(-1)^m\right]}{-\lambda_{nm}} \right] \sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{H}y}$$

If the BVP has homogs. BCs.

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{-q_{nm}}{\lambda_{nm}} \sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{H}y}$$

8.4.3

$$\begin{cases} c(x)\rho(x)\, u_t = \dfrac{\partial}{\partial x}\left(K_0(x)\dfrac{\partial u}{\partial x}\right) + q(x)u + f(x,t) & (1) \\ u(0,t) = \alpha(t), \qquad u(L,t) = \beta(t) & (2) \\ u(x,0) = g(x) & (3) \end{cases}$$

<u>Sep. Vabs:</u> (for homog. problem)

$$u(x,t) = h(t)\,\phi(x).$$

First, divide equ. (1) by $c(x)\rho(x)$.

$$\boxed{u_t = \frac{1}{c(x)\rho(x)}\frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + \frac{q(x)}{c(x)\rho(x)}u + \frac{f(x,t)}{c(x)\rho(x)}} \quad (4)$$

Thus, corresponding homog. equ. can be separated as

$$h'(t)\,\phi = \frac{1}{c\rho}\, h(t)\,\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \frac{q}{c\rho}\, h(t)\,\phi$$

Div. by $h\phi$ $\qquad\qquad L \equiv \dfrac{d}{dx}\left(K_0\dfrac{d}{dx}\right) + q.$

$$\frac{h'}{h} = \frac{1}{c\rho}\frac{1}{\phi}L\phi = -\lambda$$

<u>S-L EVP:</u>

$$\begin{cases} L\phi = -\lambda c\rho\phi & (5) \\ \phi(0) = 0, \quad \phi(L) = 0 & (6) \end{cases}$$

or $\dfrac{d}{dx}\left(\overset{0}{K_0}\dfrac{d\phi}{dx}\right) + q\phi = -\lambda \overset{0}{c\rho}\phi$

K_0, q, c, ρ conts. $\Rightarrow$ S-L. equation

Thus, there are infinitely many λ_is and corresponding ϕ_is form a complete set. In particular,

$$u(x,t) = \sum_i b_i(t)\,\phi_i(x) \qquad (7)$$

Assuming that diff term by term is valid for "t" variable

$$u_t = \sum_i b_i'(t)\,\phi_i(x) \qquad (8)$$

Also, $f(x,t)/c(x)\rho(x)$ can be expanded in terms of the eigenfunctions

$$\frac{f(x,t)}{c(x)\rho(x)} = \sum_i f_i(t)\,\phi_i(x) \qquad (9)$$

where
$$f_i(t) = \frac{\displaystyle\int_0^L \frac{f(x,t)}{\cancel{c(x)\rho(x)}}\,\phi_i(x)\,\cancel{c(x)\rho(x)}\,dx}{\displaystyle\int_0^L \phi_i^2(x)\,c(x)\rho(x)\,dx}$$

Now, equ. (4) can be written as

$$\boxed{u_t(x,t) = \frac{1}{c(x)\rho(x)} L u(x,t) + \frac{f(x,t)}{c(x)\rho(x)}} \qquad (4.2)$$

Assuming, $Lu(x,t)$ is p.w.s. in the "x" variable

$$\frac{1}{c(x)\rho(x)} Lu(x,t) = \sum_i d_i(t)\, \phi_i$$

where
$$d_i(t) = \frac{\displaystyle\int_0^L \frac{\phi_i Lu}{\cancel{c(x)\rho(x)}} \cancel{c(x)\rho(x)}\, dx}{\displaystyle\int_0^L \phi_i^2(x)\, c(x)\rho(x)\, dx} \qquad (10).$$

Subst. of (8), (9), and (10) into (4.2)

$$\sum_i b_i'(t)\, \phi_i(x) = \sum_i d_i(t)\, \phi_i + \sum_i f_i(t)\, \phi_i$$

Applying orthog.

$$b_i'(t) = d_i(t) + f_i(t) \Rightarrow \qquad (11)$$

$$\boxed{b_i'(t) = \frac{\displaystyle\int_0^L \phi_i\, Lu(x,t)\, dx}{\displaystyle\int_0^L \phi_i^2(x)\, c(x)\rho(x)\, dx} dx + f_i(t)} \qquad (12)$$

Now, using Green's formula
$$\int_0^L \phi_i\, Lu(x,t)\, dx = \int_0^L u L\phi_i\, dx + K_0(x)\left[\cancelto{0}{\phi_i \frac{\partial u}{\partial x}(x,t)} - u(x,t)\,\phi_i'(x)\right]\Big|_0^L$$

$$\Rightarrow \int_0^L \phi_i(x)\, Lu(x,t)\, dx = -\lambda_i \int_0^L u(x,t)\, \phi_i(x)\, c(x)\rho(x)\, dx$$
$$- K_0(L)\left(u(L,t)\, \phi_i'(L) + K(0)\, u(0,t)\, \phi_i'(0)\right)$$

Subst. into (12)

$$b_i'(t) = -\lambda_i \underbrace{\frac{\int_0^L u(x,t)\,\phi_i(x)\,c(x)\rho(x)\,dx}{\int_0^L \phi_i^2(x)\,c(x)\rho(x)\,dx}}_{=\,b_i(t)}$$

$$+ \frac{K_0(L)\,\overset{=\beta(t)}{u(L,t)}\,\phi_i'(L) - K_0(0)\,\overset{=\alpha(t)}{u(0,t)}\,\phi_i'(0)}{\int_0^L \phi_i^2(x)\,c(x)\rho(x)\,dx}$$

$$\Rightarrow \quad b_i'(t) + \lambda_i b_i(t) = \underbrace{\frac{K_0(0)\,\alpha(t)\,\phi_i'(0) - K_0(L)\,\beta(t)\,\phi_i'(L)}{\int_0^L \phi_i^2(x)\,c(x)\rho(x)\,dx}}_{=\,B(t)}$$

or $\boxed{b_i'(t) + \lambda_i b_i(t) = B(t)}$ (13)

From I.C.

$$g(x) = u(x,0) = \sum_i b_i(0)\,\phi_i(x)$$

$$\Rightarrow \quad b_i(0) = \frac{\int_0^L g(x)\,\phi_i(x)\,c(x)\rho(x)\,dx}{\int_0^L \phi_i^2(x)\,c(x)\rho(x)\,dx} \equiv g_i$$

$\Rightarrow$ $\boxed{b_i(0) = g_i}$ (14)

First order Linear ODE nonhomog.