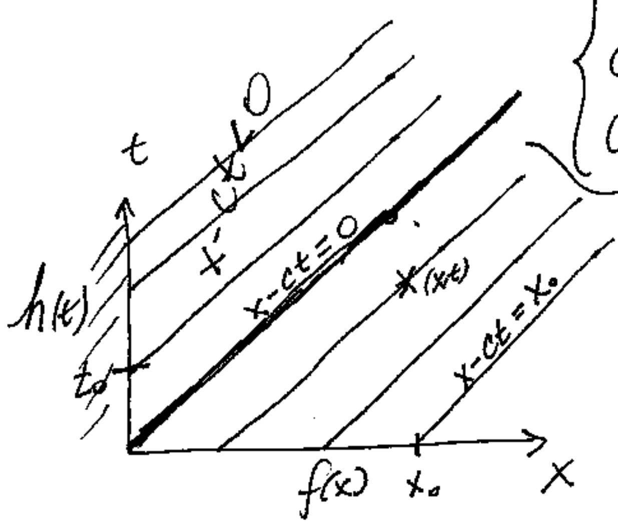
\# 12.2.4 Solve IVBVP:

$$\begin{cases} w_t + c w_x = 0, & c>0 \\ w(x,0) = f(x) & x>0,\ t>0 \\ w(0,t) = h(t) \end{cases}$$



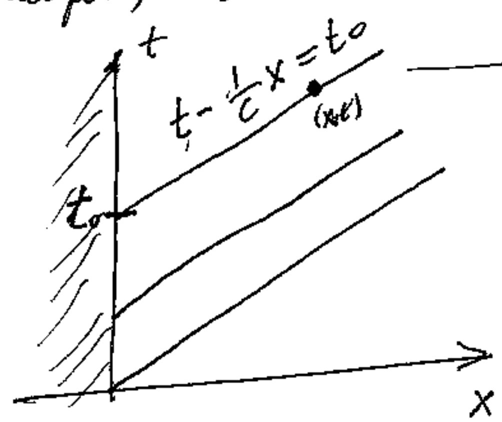
The Solution is defined by two pieces.:

a) For $x-ct>0$, the Soln. is as derived in class

$$\boxed{w(x,t) = f(x-ct)} \qquad (1)$$

b) For $x-ct<0$, the important consideration is that the information is emanating from boundary $\underline{x=0}$

Therefore, we write the characteristics as



$\longrightarrow$ So when $x=0$ $t=t_0$.

Then, $\boxed{w(x,t) = h\left(t-\frac{1}{c}x\right)}$ (2)

Since, clearly

$w(0,t) = h(t)$ ✓

and

$$w_t + cw_x = h'\left(t-\frac{1}{c}x\right) + c\left(-\frac{1}{c}\right)h'\left(t-\frac{1}{c}x\right)$$

$$= 0 \checkmark$$

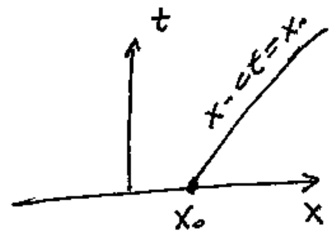
12.2.5 a) Solve: $\begin{cases} w_t + c w_x = e^{2x} & (1) \\ w(x,0) = f(x). & (2) \end{cases}$

- Assume $w(x,t)$ is a Soln. of (1)-(2).
- Consider a curve $x = x(t)$
- Define $\hat{w}(t) = w(x(t),t)$. $w(x,t)$ along curve $x = x(t)$.

Then, $\frac{d\hat{w}}{dt}(t) = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial t}$.

If $\boxed{\frac{dx}{dt}(t) = c}$ (3)

then $\boxed{\frac{d\hat{w}}{dt}(t) = \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x(t)}}$ (4)



From (3) $x(t) = ct + x_0$

or $\boxed{x_0 = x - ct}$

Subst. into (4) $\frac{d\hat{w}}{dt}(t) = e^{2(ct + x_0)} = e^{2x_0} e^{2ct}$

$\Rightarrow \hat{w}(t) = \frac{e^{2x_0} e^{2ct}}{2c} + D$

Now, $\hat{w}(0) = w(x(0),0) = w(x_0,0) = f(x_0)$

$\Rightarrow D + \frac{e^{2x_0} e^{0}}{2c} = f(x_0) \Rightarrow D = f(x_0) - \frac{e^{2x_0}}{2c}$

$$\therefore \quad \hat{w}(t) = \frac{1}{2c}\left[e^{2(ct+x_0)} + 2c f(x_0) - e^{2x_0} \right]$$

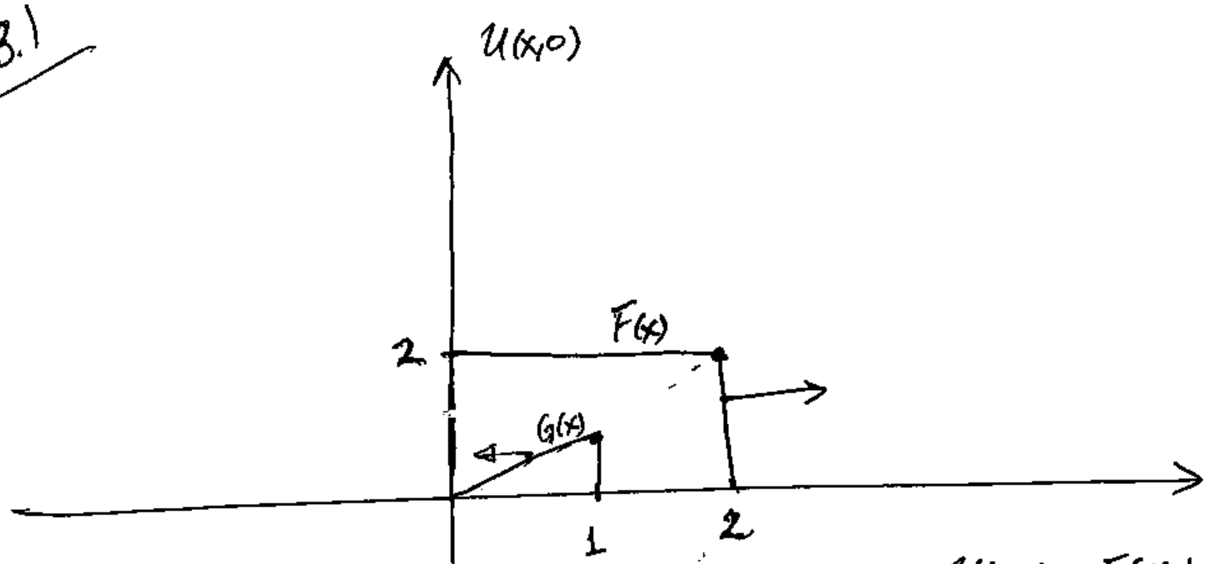
$$\Rightarrow \quad w(x(t),t) = \hat{w}(t) = \frac{1}{2c}\left[e^{2x(t)} + 2c f(x-ct) - e^{2(x-ct)} \right]$$

Since the characteristics : $x - ct = x_0$ fill the semi-plane

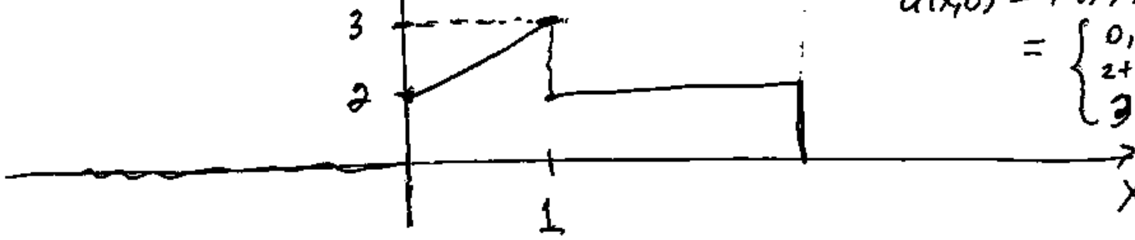
The soln. for the original IVP. is given by

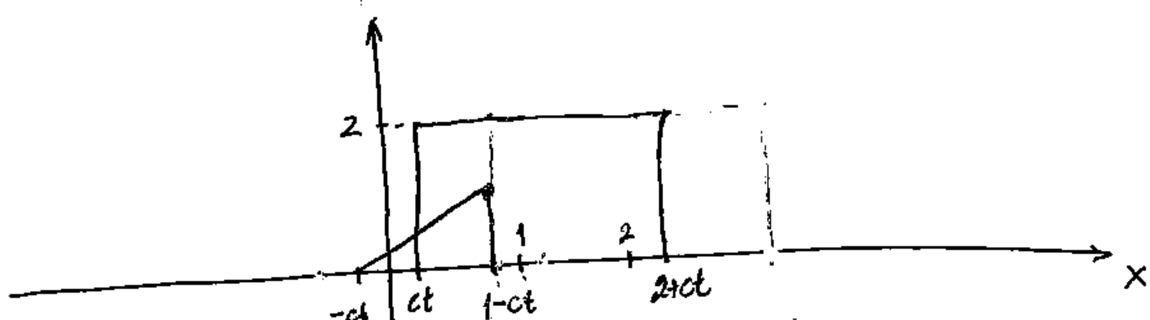
$$\boxed{w(x,t) = \frac{1}{2c}\left[e^{2x} - e^{2(x-ct)} + 2c f(x-ct) \right]}$$

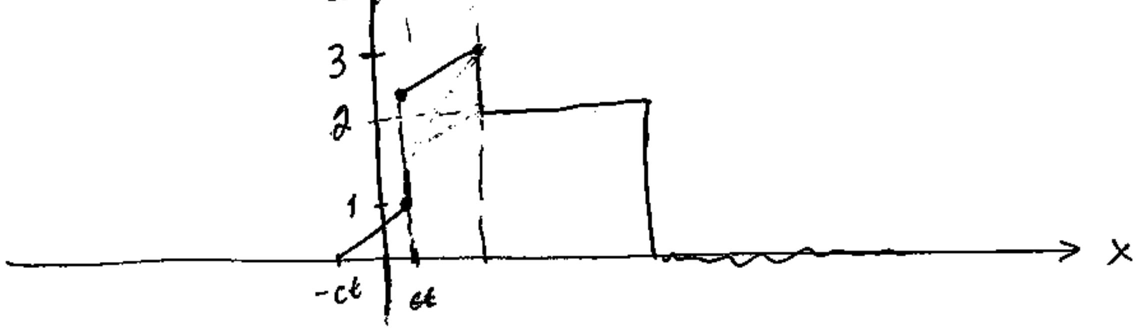
#12.3.1

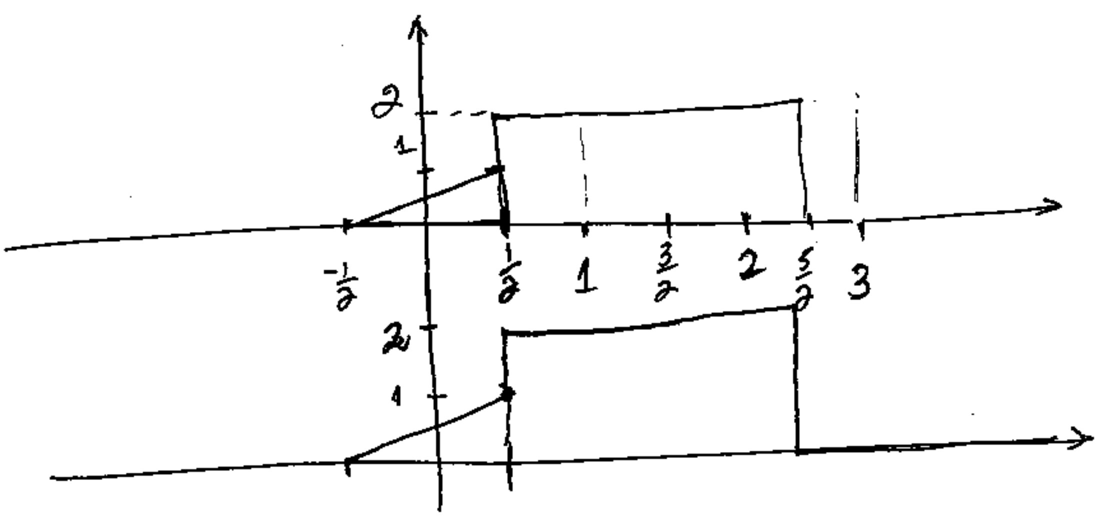


$$U(x,0) = F(x) + G(x) = \begin{cases} 0, & x<0 \\ 2+x, & 0<x<1 \\ 2, & 1<x<2 \\ 0, & x>2 \end{cases}$$









12.3.5

Example. Solve the IVP.

$$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \quad t > 0.$$

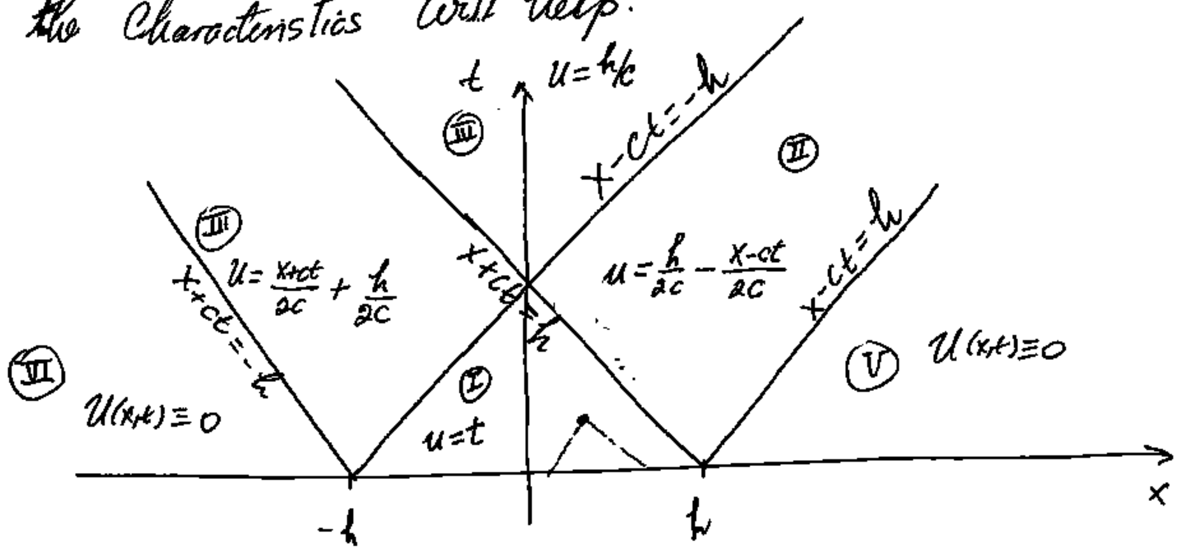
$$u(x,0) = f(x) \equiv 0, \qquad u_t(x,0) = \begin{cases} 1, & |x| < h \\ 0, & |x| > h. \end{cases} \equiv g(x)$$

D'Alambert formula

$$u(x,t) = \frac{1}{2}\left[\cancel{f(x-ct)}^{\,0} + \cancel{f(x+ct)}^{\,0} \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}}$$

To obtain $u(x,t)$ a graphic of the xt-plane and the Characteristics will help.



In Region (I): $-h < x-ct$ and $x+ct < h$

$$\Rightarrow u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_{x-ct}^{x+ct} 1\,d\bar{x} = \frac{1}{2c}[x+ct-x+ct] = \frac{2ct}{2c} = t.$$

$$\Rightarrow \boxed{u(x,t) = t}$$

In Region II: $-h < x-ct < h$ and $x+ct > h$.

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_{x-ct}^{h} \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} + \frac{1}{2c}\int_{h}^{x+ct} 0\,d\bar{x}.$$

$$\therefore \quad \boxed{u(x,t) = \frac{h-x+ct}{2c} = -\frac{(x-ct)}{2c} + \frac{h}{2c}}$$

In Region IV: $x-ct < -h$ and $x+ct > h$.

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_{x-ct}^{-h} \overset{\equiv 0}{g(\bar{x})}\,d\bar{x} + \frac{1}{2c}\int_{-h}^{h} \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} + \frac{1}{2c}\int_{h}^{x+ct} \overset{\equiv 0}{g(\bar{x})}\,d\bar{x}$$

$$\therefore \quad \boxed{u(x,t) = \frac{h-(-h)}{2c} = \frac{h}{c}}$$

In Region III: $x-ct < -h$ and $-h < x+ct < h$.

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \overset{\equiv 0}{g(\bar{x})}\,d\bar{x} = \frac{1}{2c}\int_{x-ct}^{-h} \overset{\equiv 0}{g(\bar{x})}\,d\bar{x} + \frac{1}{2c}\int_{-h}^{x+ct} \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} +$$

$$\therefore \quad \boxed{u(x,t) = \frac{1}{2c}\left[\frac{x+ct}{2c} + \frac{h}{2c}\right] = \frac{(x+ct)}{2c} + \frac{h}{2c}}$$

In Region V: $x-ct > h$ and $x+ct > h$

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \overset{=0}{g(\bar{x})}\,d\bar{x} = 0. \quad \Rightarrow \quad \boxed{u(x,t) \equiv 0.}$$

In Region VI: $x-ct < -h$ and $x+ct < -h$

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \overset{=0}{g(\bar{x})}\,d\bar{x} = 0 \quad \Rightarrow \quad \boxed{u(x,t) \equiv 0}$$

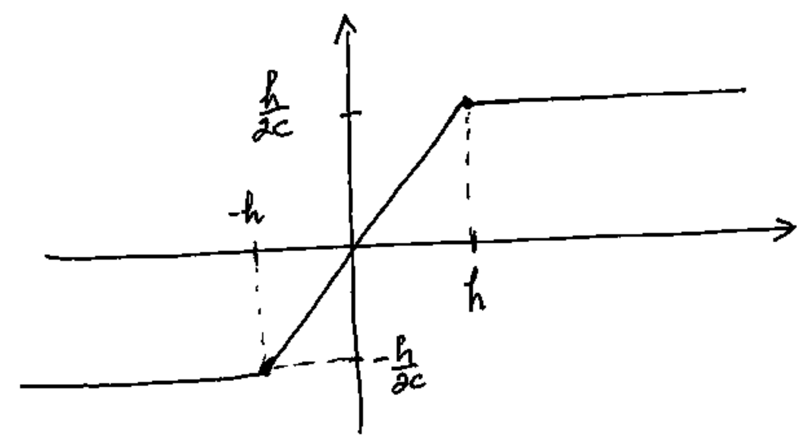
Graphing the time evolution of the wave equation for our last example.

1) Graph of

$$G(x)=\frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} = \frac{1}{2c}\begin{cases} \text{If } x<-h, & \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} = -\frac{1}{2c}\int_x^{-h} \overset{=0}{g(\bar{x})}\,d\bar{x} - \frac{1}{2c}\int_{-h}^0 \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} = \frac{-h}{2c} \\ \text{If } -h<x<0, & \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} = -\frac{1}{2c}\int_x^0 \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} = \underline{\underline{\frac{1}{2c}}}\,x \\ \text{If } 0<x<+h, & \frac{1}{2c}\int_0^x \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} = \frac{1}{2c}x. \\ \text{If } x>h, & \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_0^h \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} + \frac{1}{2c}\int_h^x \overset{\equiv 0}{g(\bar{x})}\,d\bar{x} = \frac{h}{2c} \end{cases}$$

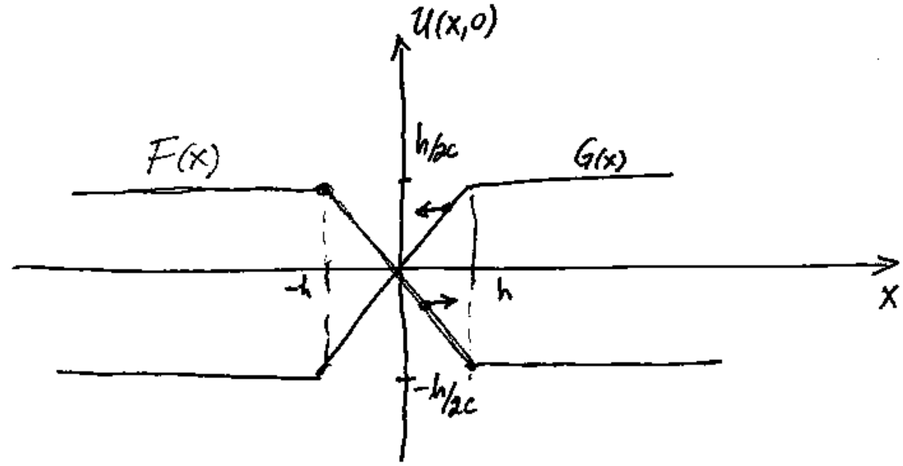
Summarizing:

$$G(x) = \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} = \begin{cases} -\frac{h}{2c}, & x<-h \\ \frac{x}{2c}, & -h<x<h. \\ \frac{h}{2c}, & x>h. \end{cases}$$



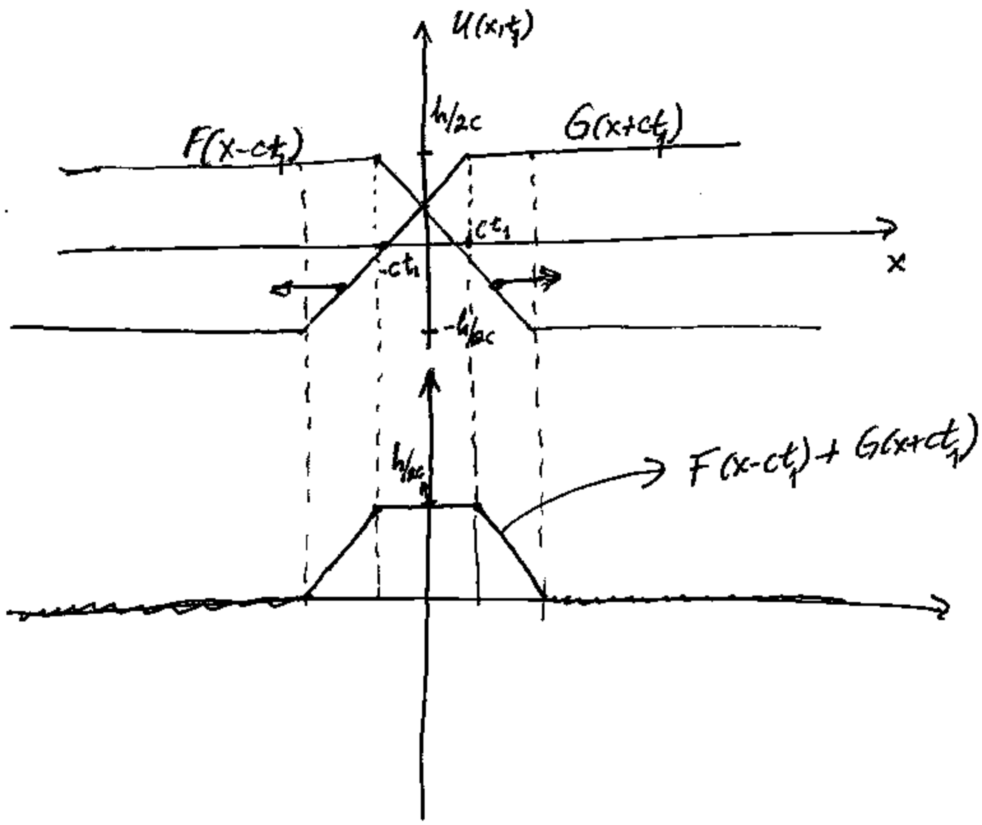
2) Form $F(x)$ and $G(x)$.

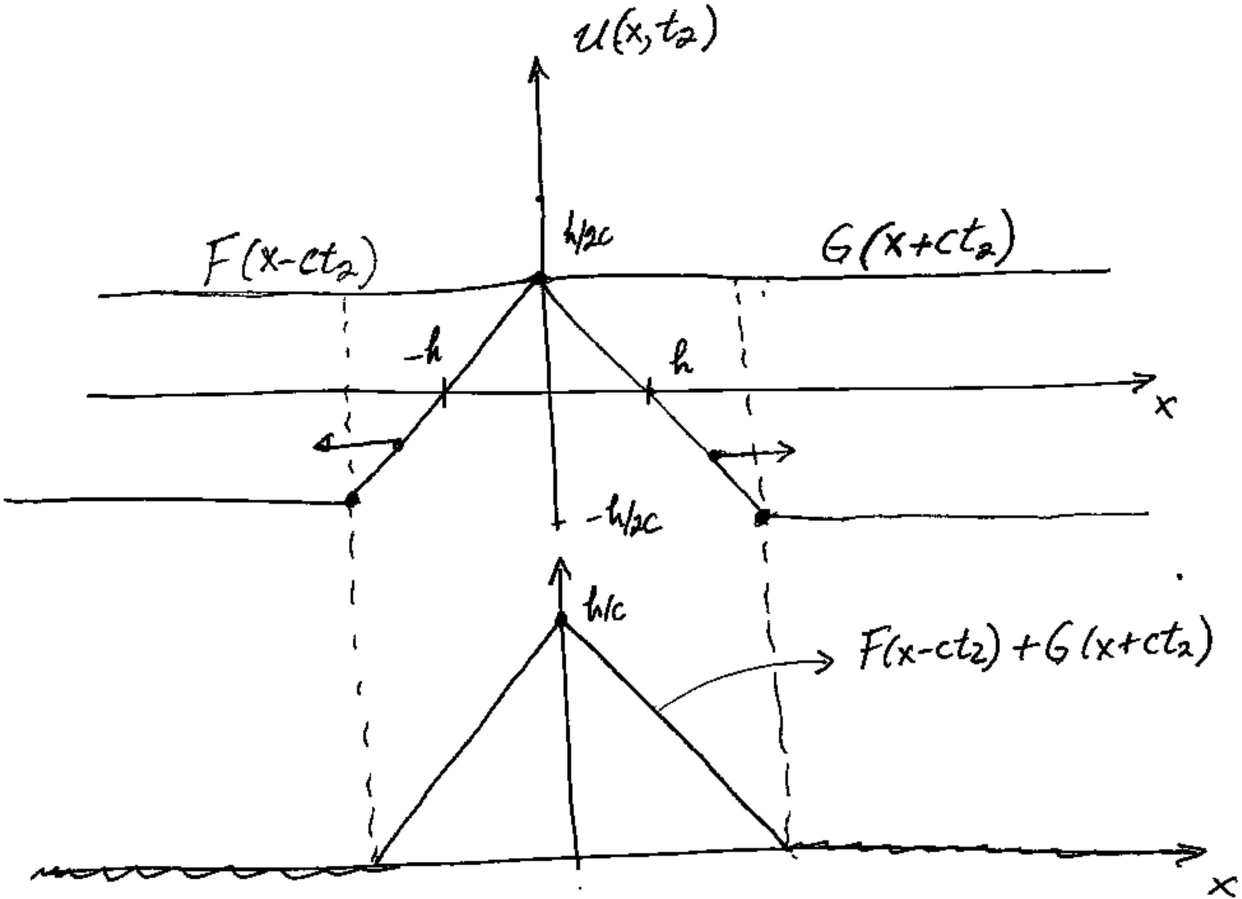
$$F(x) = -\frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x}, \qquad G(x) = \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x}$$



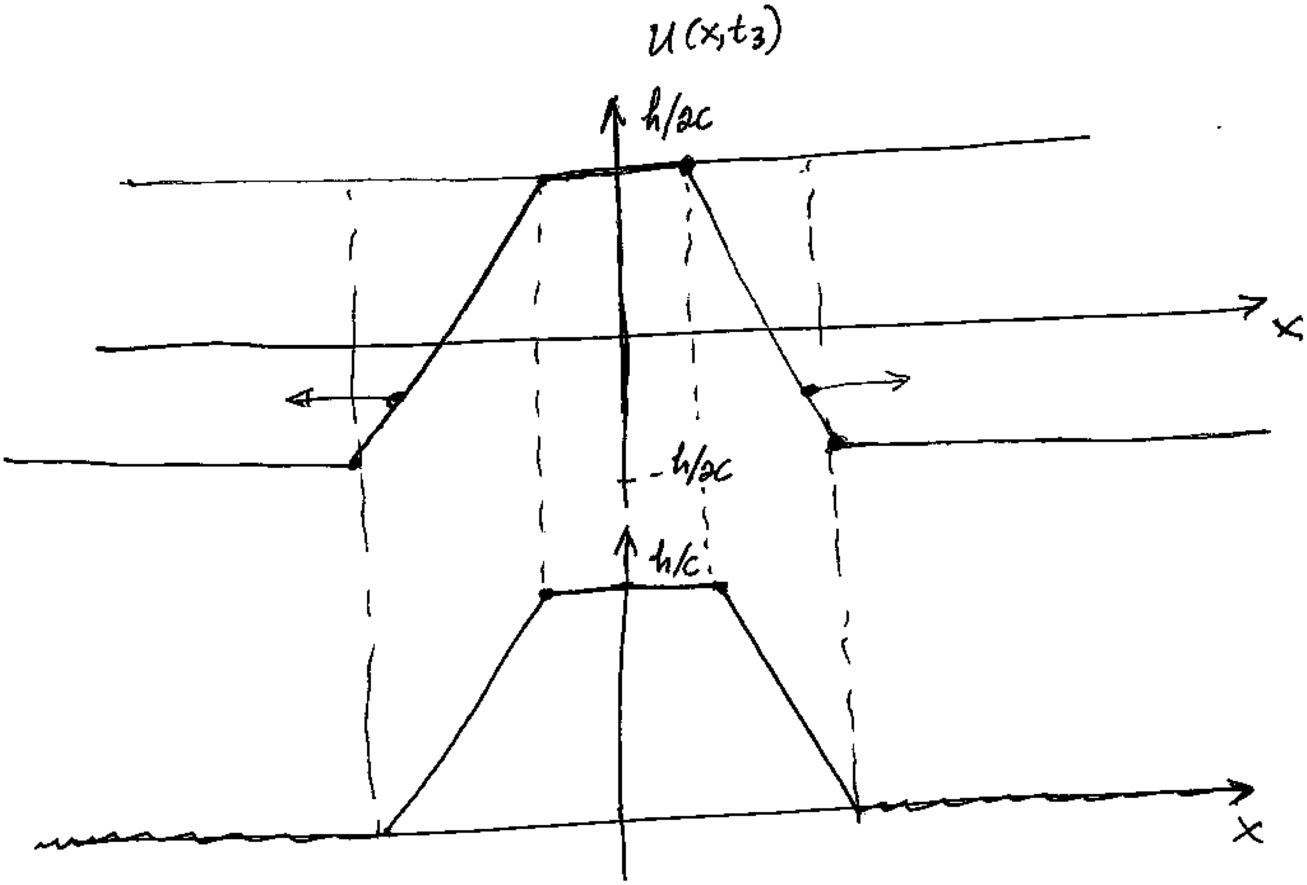
$$u(x,0) = F(x) + G(x) \equiv 0. \quad \text{for all } x$$

3), 4) Translate $F(x)$ to the right $\to F(x-ct)$ } $\Rightarrow$ Add the two functions
Translate $G(x)$ to the left $\to G(x+ct)$. } to obtain $u(x,t) = F(x-ct) + G(x+ct)$





u(x, t2)
h/2c
F(x-ct2)
G(x+ct2)
-h
h
x
-h/2c
h/c
F(x-ct2) + G(x+ct2)
x



u(x, t3)
h/2c
x
-h/2c
h/c
x

#12.3.6 $$u_{tt} = \frac{c^2}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right)$$

$$u(\rho,t) = \frac{1}{\rho}\,w(\rho,t).$$

$$\Rightarrow\ u_{tt} = \frac{1}{\rho}w_{tt}$$

$$\frac{\partial u}{\partial\rho} = -\frac{1}{\rho^2}w + \frac{1}{\rho}w_\rho$$

$$\Rightarrow 0 = u_{tt} - \frac{c^2}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) =$$

$$= \frac{1}{\rho}w_{tt} - \frac{c^2}{\rho^2}\frac{\partial}{\partial\rho}\left(-w+\rho w_\rho\right)$$

$$\Rightarrow \quad \frac{1}{\rho}w_{tt} = \frac{c^2}{\rho^2}\left[\cancel{-w_\rho} + \cancel{w_\rho} + \rho w_{\rho\rho}\right]$$

$$\Rightarrow \boxed{w_{tt} = c^2 w_{\rho\rho}},$$

$$\Rightarrow\ w(\rho,t) = F(\rho-ct) + G(\rho+ct).$$

$$\Rightarrow\ u(\rho,t) = \frac{F(\rho-ct)}{\rho} + \frac{G(\rho+ct)}{\rho}$$ amplitude is decaying as $\frac{1}{\rho}$.

12.4.1 (More general)

$$\begin{cases} u_t = c^2 u_{xx}, & x>0,\ t>0 & (1) \\ u(x,0) = f(x), & x>0 & (2) \\ u_t(x,0) = g(x), & x>0 & (3) \\ u(0,t) = h(t), & t>0 & (4) \end{cases}$$

General Soln. of (1): $\boxed{u(x,t) = F(x-ct) + G(x+ct)}$ (6.1)

Using IC's is possible to determine F and G for $x>0$.

In fact,

$$f(x) = u(x,0) = F(x) + G(x), \quad x>0 \qquad (5)$$

$$g(x) = u_t(x,0) = -cF'(x) + cG'(x), \quad x>0 \qquad (6)$$

$$\Rightarrow f'(x) = F'(x) + G'(x)$$

$$\Rightarrow f'(x) + \frac{1}{c} g(x) = 2G'(x) \Rightarrow$$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\bar{x})d\bar{x} + K$$

From (5) $F(x) = f(x) - G(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\bar{x})d\bar{x} - K.$

Summarizing,

$$\boxed{\begin{aligned} F(x) &= \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\bar{x})d\bar{x} - K, \quad x>0 & (5.2) \\ G(x) &= \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\bar{x})d\bar{x} + K, \quad x>0 & (5.3) \end{aligned}}$$

Therefore, the Soln. of (1)-(4) is Completely Known when

(I) $\underline{\underline{x-ct>0}}$

$$u(x,t) = F(x-ct)+G(x+ct) =$$
$$= \frac{1}{2}\left[f(x-ct)+f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} \quad (7)$$

However, when

(II) $\underline{x-ct<0}$

$F(x-ct)$ is not defined.

To obtain the Soln. in this region, we use (5.1) and (4).

$$h(t) = u(0,t) = F(-ct) + G(ct) \quad (8)$$

$F(-ct)$ is not defined. However, from (8)

$$F(-ct) = h(t) - G(ct)$$

if $z=-ct$ then, $ct=-z$, and $t=-\frac{z}{c}$

Thus, $$F(z) = h\left(-\frac{z}{c}\right) - G(-z), \quad z<0 \quad (9)$$

Equ. (9) represents the def. of F for negative numbers

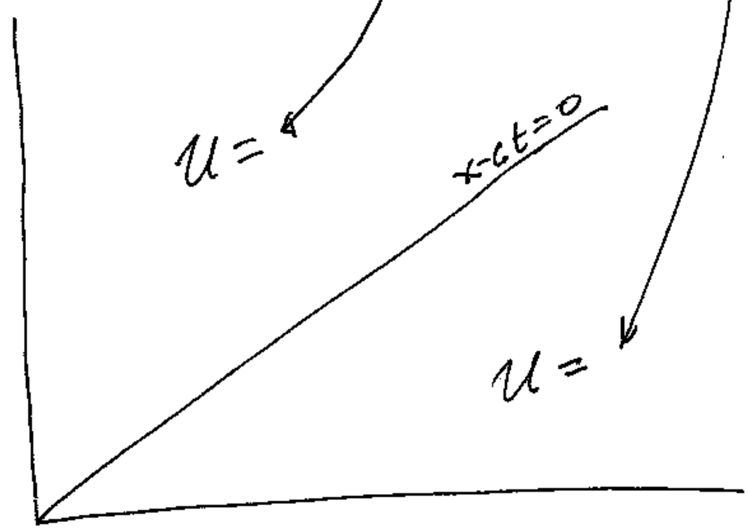
Therefore, the solution $u(x,t)$ when $\underline{x-ct<0}$ can be written as

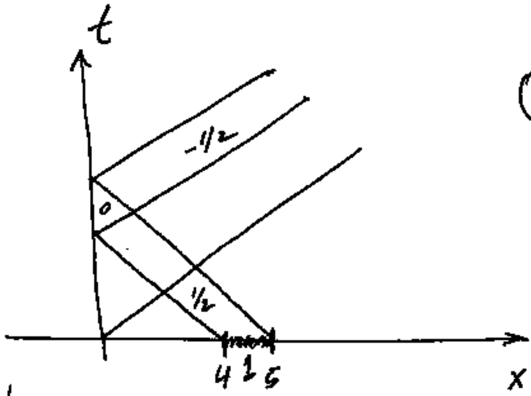
$$u(x,t) = F(x-ct)+G(x+ct) = h\left(\frac{ct-x}{c}\right) - \frac{1}{2}f(ct-x) - \frac{1}{2c}\int_0^{ct-x} g(\bar{x})\,d\bar{x}$$
$$+ \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(\bar{x})\,d\bar{x}$$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\bar{x})\,d\bar{x} + h\left(t-\frac{x}{c}\right)} \quad (10)$$

Therefore, Solution is given by

$$u(x,t) = \begin{cases} \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}, & x-ct>0. \\ \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\bar{x})\,d\bar{x}, & x+ct<0 \end{cases}$$





(I) If $x < ct$ and Dirichlet B.C. $u(0,t)=0$

$u(x,t) = G(x+ct) - G(ct-x)$.

$$\boxed{G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x}} \quad (*)$$

Similar to

12.4.3

(II) If $x < ct$ and $0 < x+ct$ and Neumann B.C. $u_x(0,t)=0$

$$\Rightarrow \boxed{u(x,t) = F(x\overset{?}{-}ct) + G(x+ct)}$$

$$u_x(x,t) = F'(x-ct) + G'(x+ct)$$

$$0 = u_x(0,t) = F'(-ct) + G'(ct).$$

$$\Rightarrow \text{if } y = -ct \qquad F'(y) = -G'(-y).$$

$$\Rightarrow \quad F(y) = G(-y) + H$$

$H=0$ for $u(x,t)$ to be conts at $(x,t)=(0,0)$

(Hint: Use the two formulas: the one below $x-ct=0$ ($x>ct$) and the one above $x-ct=0$ ($x<ct$) and evaluate them at $(0,0)$.)

I think H may not be zero if there is a jump discont. at $(0,0)$. However, in book they add the cond. of continuity at $(0,0)$.

$$\Rightarrow u(x,t) = G(ct-x) + G(x+ct).$$

In region $x-ct<0$ and $x+ct>0$.

Using the formula (*) for $G(x)$.

$$u(x,t) = \frac{1}{2}\left[f(ct-x) + f(x+ct)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\bar{x})\,d\bar{x}.$$

If $g(x) \equiv 0$ and $f(x) = \begin{cases} 1, & 0<x<5 \\ 0, & \text{otherwise} \end{cases}$

If $G(x) = \frac{1}{2} f(x)$. only

If $x - ct < 0$.

$\Rightarrow \quad u(x,t) = \frac{1}{2}\left[f(ct-x) + f(x+ct) \right]$

<u>Two Procedures:</u>

(I) Find $x - ct < 0$ and take the symmetric point $ct - x > 0$.

Determine where $ct - x$ is located. If $ct - x \notin [a,b]$

$\Rightarrow \quad f(ct-x) = 0$, otherwise $f(ct-x) \neq 0$.

and $u(x,t) = \frac{1}{2}[\quad]$

(II) Construct the even extension of $f(x)$ around $x=0$ on the real line.

i.e., $\quad \hat{f}(x) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$

$\Rightarrow$ If $x - ct < 0$

$$u(x,t) = \frac{1}{2}\left[\hat{f}(x-ct) + \hat{f}(x+ct) \right] =$$

$$= \frac{1}{2}\left[f(ct-x) + f(x+ct) \right]$$

<u>Same as before.</u>

12.4.4 Same as 12.4.1 (general) but B.C.: $u_x(0,t) = j(t)$.

$$\begin{cases} u_t = \kappa u_{xx}, & x>0,\ t>0 \quad (1) \\ u(x,0) = f(x), & x>0 \quad (2) \\ u_t(x,0) = g(x), & x>0 \quad (3) \\ u_x(0,t) = j(t), & t>0 \quad (4) \end{cases}$$

General Soln: $u(x,t) = F(x-ct) + G(x+ct)$. (5)

From IC's.
$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x}$$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x}$$

the constant K can be omitted because it will cancel when F and G are added.

Again (as in 12.4.1) the Soln in region is given by (5)

$\underline{\underline{x-ct>0}}$

$$u(x,t) = F(x-ct) + G(x+ct) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$$

On Region:

$\underline{\underline{x-ct<0}}$ $F(x-ct)$ is not defined

we want to use B.C. (4). So we need

$$u_x(x,t) = F'(x-ct) + G'(x+ct)$$

and

$$j(t) = u_x(0,t) = F'(-ct) + G'(ct), \quad t>0$$

$$\Rightarrow F'(-ct) = j(t) - G'(ct)$$

then, $z=-ct \Rightarrow t=\frac{-z}{c}$, $ct=-z$

$$F'(z) = g\left(-\frac{z}{c}\right) - G'(-z)$$

$\bar{u} = -\frac{\bar{z}}{c}$

$d\bar{u} = -\frac{1}{c} d\bar{z}$

$$\Rightarrow \int_0^z F'(\bar{z})\,d\bar{z} = \int_{\bar{z}=0}^{\bar{z}=z} g\left(-\frac{\bar{z}}{c}\right) d\bar{z} - \int_0^z G'(-\bar{z})\,d\bar{z}$$

$\int G'(-z)\,dz =$

$u=-z \Rightarrow du=-dz$

$-\int G'(u)\,du = -G(u)+c$.

$\int_0^z g\left(-\frac{\bar{z}}{c}\right) d\bar{z} = -c\int_{\bar{u}=0}^{\bar{u}=-z/c} g(\bar{u})\,d\bar{u}$

$$\Rightarrow F(z) - F(0) = \int_0^{\bar{u}=-z/c} g(\bar{u})\,d\bar{u} + G(-z) - G(0)$$

$$\Rightarrow \boxed{F(z) = G(-z) + \int_0^{\bar{u}=-z/c} g(\bar{u})\,d\bar{u} + F(0) - G(0), \quad z<0}$$

Definition of F for negative numbers.

there, soln on region $\underline{\underline{x-ct<0}}$

$$u(x,t) = F(x-ct) + G(x+ct) = G(ct-x) + G(x+ct) + \int_0^{t-x/c} g(\bar{u})\,d\bar{u} + K$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[f(x+ct) + f(ct-x)\right] + \frac{1}{2c}\int_0^{ct-x} g(\bar{x})\,d\bar{x} + \frac{1}{2c}\int_0^{x+ct} g(\bar{x})\,d\bar{x}$$

$$+ \int_0^{t-x/c} g(\bar{u})\,d\bar{u} + K \text{ (arbitr. constant)}.$$

Summarizing:

$$
u(x,t) = \begin{cases} \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}, & x-ct>0. \\[2ex] \frac{1}{2}\left[f(x-ct)+f(ct-x)\right] + \frac{1}{2c}\int_0^{ct-x} g(\bar{x})\,d\bar{x} + \frac{1}{2c}\int_0^{x+ct} g(\bar{x})\,d\bar{x} \\ \qquad + \int_0^{t-x/c} j(\bar{u})\,d\bar{u} + K, & x-ct<0. \end{cases}
$$

Alternative procedure in Region $x-ct<0$ for B.C. $\underline{\underline{u(0,t)=0}}$

Odd extension of $f(x)$ to the infinite real line.

$$\hat{f}(x) = \begin{cases} f(x), & x>0 \\ -f(-x), & x<0 \end{cases}$$

$\Rightarrow$ If $x-ct<0$

$$u(x,t) = \frac{1}{2}\left[\hat{f}(x-ct) + \hat{f}(x+ct)\right] =$$

$$= \frac{1}{2}\left[-f(ct-x) + f(x+ct)\right]$$ Same as before.

Problem #12.4.3

If B.C. $u_x(0,t)=0$ and $x-ct<0$.

$$u(x,t) = F(x-ct) + G(x+ct).$$

$$u_x(x,t) = F'(x-ct) + G'(x+ct).$$

$$\Rightarrow 0 = u_x(0,t) = F'(-ct) + G'(ct)$$

$$\Rightarrow F'(y) = -G'(-y) \Rightarrow F(y) = G(-y)$$ (Constant of integration is zero see next page)

$$\Rightarrow F(x-ct) = G(ct-x).$$

and the soln. in this region

$$\boxed{u(x,t) = G(ct-x) + G(x+ct).}$$

Problems Section 12.5.

12.5.1

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x), \quad u_t(x,0) = g(x). \\ u(0,t) = 0, \quad u(L,t) = 0 \end{cases}$$

a) $$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \tfrac{n\pi}{L}ct \, \sin\left(\tfrac{n\pi}{L}x\right) + B_n \sin \tfrac{n\pi}{L}ct \, \sin \tfrac{n\pi}{L}x \right]$$

where

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \tfrac{n\pi}{L}x \, dx$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin \tfrac{n\pi}{L}x \, dx.$$

If $g(x) = 0$.

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\tfrac{n\pi}{L}ct\right) \sin\left(\tfrac{n\pi}{L}x\right)$$

$$\sin\left(\tfrac{n\pi}{L}(x-ct)\right) - \sin\left(\tfrac{n\pi}{L}(x+ct)\right)$$

$$= 2 \sin\left(\tfrac{n\pi}{L}x\right) \cos\left(\tfrac{n\pi}{L}ct\right)$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} A_n \left[\sin\left(\tfrac{n\pi}{L}(x-ct)\right) + \sin\left(\tfrac{n\pi}{L}(x+ct)\right) \right]$$

$\Rightarrow$

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}(x-ct)\right) + \frac{1}{2} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}(x+ct)\right)$$

Then

$$f(x) = u(x,0) = \frac{1}{2}\Big[\underbrace{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)}_{\mathrm{I}} + \underbrace{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)}_{\mathrm{II}}\Big] = \underbrace{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)}_{\mathrm{IV}}$$

where

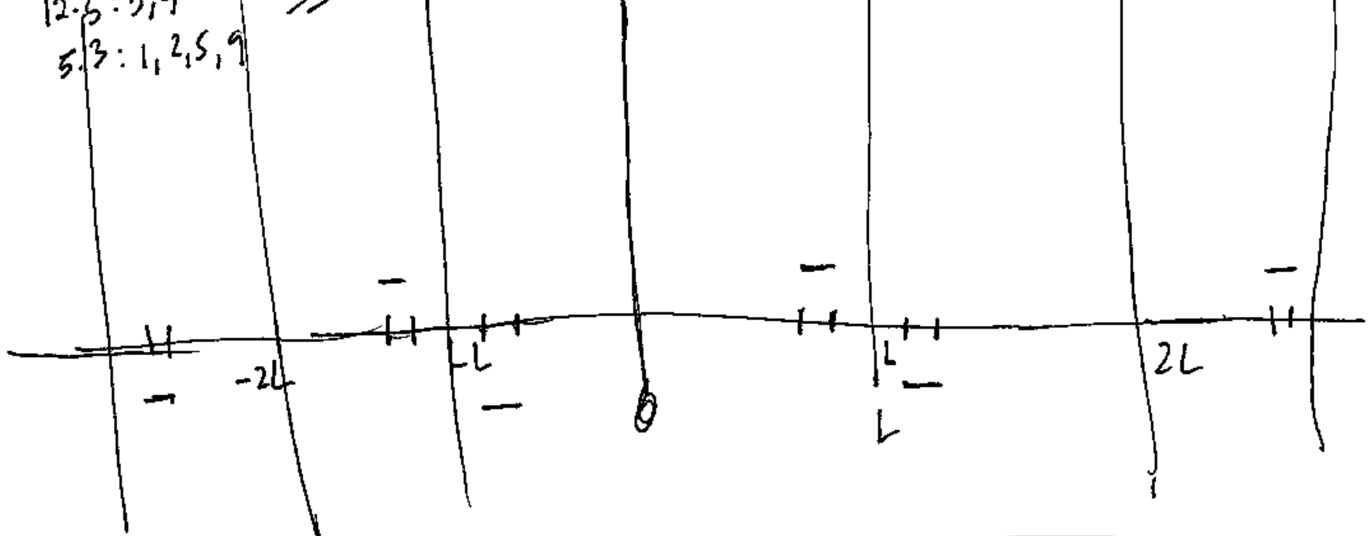
$$\underbrace{A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx.}_{\mathrm{III}}$$

The pair IV, III represent the odd periodic (period $= 2L$) extension $\underline{\underline{f_{ext}(x)}}$ of $f(x)$. It means $f_{ext}(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$, A_n given by III.

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2} f_{ext}(x-ct) + \frac{1}{2} f_{ext}(x+ct)}$$

12.4: 1-4.
12.5: 3,4
5.3: 1,2,5,9

#12.5.3



12.5.3