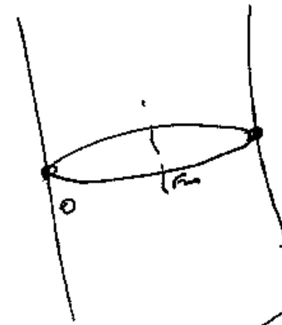
4.4.1 a) $\omega = \frac{n\pi}{L} c.$



$\sin\left(m-\frac{1}{2}\right)\frac{\pi c}{H} t$

$\frac{L}{2}$

If $m=1$

$$\sin\left(m-\frac{1}{2}\right)\frac{\pi x}{H} = \sin\left(\frac{\pi}{2H}x\right) = \sin\frac{\pi}{L}x \quad \text{if } H=\frac{L}{2}$$

If $m=2$

$$\sin\left(m-\frac{1}{2}\right)\frac{\pi x}{H} = \sin\left(\frac{3}{2}\frac{\pi}{H}x\right) = \sin\left(\frac{3}{2}\frac{\pi}{\frac{L}{2}}x\right) =$$
$$= \sin\left(\frac{3\pi}{L}x\right).$$

4.4.2 b) $\rho_0(x)\, u_{tt} = T_0\, u_{xx} + \alpha(x) u.$

$u(x,t) = \phi(x) h(t)$

$$\frac{\rho_0(x)\, h''\phi = T_0 h(t)\phi'' + \alpha(x)\phi h}{\rho_0(x) h(t)\phi(x)}$$

$$\cancel{\frac{\rho_0(x) h''\phi - T_0 h(t)\phi'' = \alpha(x)\phi h}{h\phi\alpha(x)}}$$

$$\frac{h''(t)}{h} = \frac{T_0}{\rho_0(x)}\frac{\phi''}{\phi} + \frac{\alpha(x)}{\rho_0(x)} = -\lambda$$

$$\boxed{h''(t) + \lambda h(t) = 0} \qquad \frac{\phi''}{\phi} + \frac{\alpha(x)}{T_0} = \frac{\rho_0(x)}{T_0}\lambda$$

2

$$\phi'' + \frac{1}{T_0}\left(\alpha(x) + \lambda\rho_0(x)\right)\phi = 0$$

or $$\phi'' + \frac{\alpha(x)}{T_0}\phi + \frac{\lambda\rho_0(x)}{T_0}\phi = 0$$

---

c) If $\alpha(x) = \alpha$, const $\rho_0(x) = \rho_0$ consts

$$\phi'' + \frac{(\alpha + \lambda\rho_0)}{T_0}\phi = 0 \Rightarrow \frac{\alpha + \lambda\rho_0}{T_0} = \left(\frac{n\pi}{L}\right)^2$$

$$\lambda = \left(\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{T_0}\right)\frac{T_0}{\rho_0}$$

$$\boxed{\lambda_n = \frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\alpha}{\rho_0}}$$

9''

In space at time fixed # 4.4.2 Haberman

$$T_0 \hat{u}_{xx} + \alpha \hat{u} = 0$$

$$\hat{u}_{xx} + \frac{\alpha}{T_0} \hat{u} = 0$$

$\underline{\alpha > 0}$ $\quad r^2 = -\frac{\alpha}{T_0} \Rightarrow r = \pm i \sqrt{\frac{\alpha}{T_0}}$

$$\hat{u}(x) = C_1 \cos\left(\sqrt{\tfrac{\alpha}{T_0}}\, x\right) + C_2 \sin\left(\sqrt{\tfrac{\alpha}{T_0}}\, x\right)$$

$my'' + \cancel{\beta y'} + ky = 0$

$[y(t)] = L$

$[my''] = m\frac{L}{t^2}$

$[ky] = m\frac{L}{t^2}$

$\Rightarrow k = \frac{m}{t^2}$

Isolating αu force from tension. at a fixed point x.

In time at fixed position x

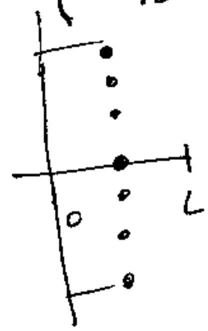
$$\rho_0 u_{tt} = \alpha u$$

$$\hat{u}_{tt} - \frac{\alpha}{\rho_0} \hat{u} = 0 \Rightarrow r^2 = \frac{\alpha^{<0}}{\rho_0} \Rightarrow r = \pm i \sqrt{\frac{-\alpha}{\rho_0}}$$

$$\text{or } \hat{u}(t) = C_1 \cos\left(\sqrt{\tfrac{-\alpha}{\rho_0}}\, t\right) + C_2 \sin\left(\sqrt{\tfrac{-\alpha}{\rho_0}}\, t\right)$$

good!

If we think in a string particle.



$$\left[\frac{\alpha}{\rho_0} u\right] = \frac{L\alpha}{\frac{M}{L}} = \frac{\alpha L^2}{M} = \frac{L}{t^2}$$

$$\Rightarrow \boxed{\alpha = \frac{M}{t^2 L}}$$

Not on web

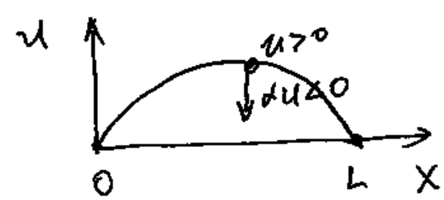
Review of Problems 4.4.2 and 4.4.3 in Habermann's book. Physical Interpretation.

4.4.2

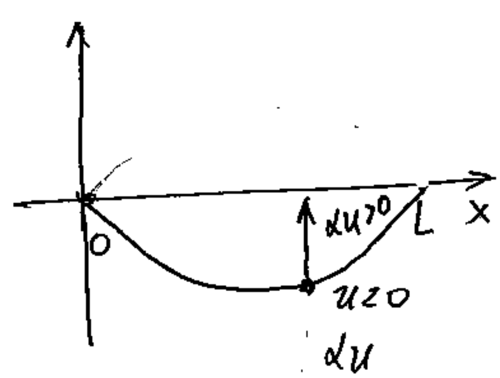
$$\rho_0 u_{tt} = T_0 u_{xx} + \alpha u$$

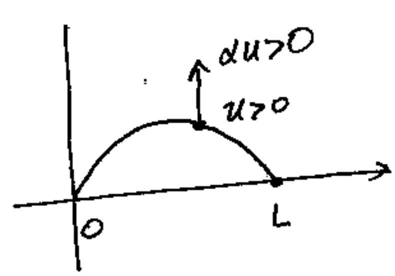
a) If $\boxed{\alpha < 0}$ RESTORING FORCE.

a1) $u(x,t) > 0 \Rightarrow \alpha u(x,t) < 0 \Rightarrow$ Negative force acting on the string push it to the equilibrium position.



a2) $u(x,t) < 0 \Rightarrow \alpha u(x,t) > 0 \Rightarrow$ Positive force acting on the string push it to the equilibrium position





b) If $\alpha > 0$

b1) $u(x,t) > 0$

$\Rightarrow \alpha u(x,t) > 0$. Force push string away from equilibrium position.

Total Energy in the Vibrating String. Haberman's problems. E_0

4.4.9

From $u_{tt} = c^2 u_{xx}$ derive

$$\frac{dE}{dt} = c^2 u_x u_t \Big|_0^L$$

Where $E(t) = KinE(t) + PotE(t) = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$

$$\frac{dE}{dt} = \int_0^L \frac{2}{2} u_t u_{tt}(x,t)\,dx + c^2 \int \frac{2}{2} u_x u_{xt}\,dx =$$

$$\underset{u_{tt}=u_{xx}c^2}{\overset{\text{Using equ.}}{=}} \int_0^L c^2 u_t u_{xx}\,dx + c^2 \int u_x u_{xt}\,dx =$$

$$= c^2 \int_0^L \left(u_t u_{xx} + u_x u_{xt}\right) dx = c^2 \int_0^L \frac{\partial}{\partial x}\left(u_t u_x\right) dx =$$

$$= c^2 u_x u_t \Big|_0^L \checkmark$$

See next page for a complete derivation from $u_{tt} = c^2 u_{xx}$

$u_{tt} = c^2 u_{xx}$ is equivalent to

$$\rho u_{tt} = T_0 u_{xx}$$

$$[\rho u_{tt}] = \frac{M}{L}\frac{L}{t^2} = \text{Force}/L$$

Multiplying by velocity u_t both members of wave equation, we obtain

$$\boxed{\rho \underbrace{u_t u_{tt}}_{\text{I}} = T_0 \underbrace{u_{xx} u_t}_{\text{II}},} \quad (*) \qquad [\rho u_t u_{tt}] = \frac{\text{Force}}{L} \cdot \underbrace{\left(\frac{L}{t}\right)}_{\text{Velocity}} = \text{Energy}/(L * t)$$

Force * L

$$\text{I} = \frac{\partial}{\partial t}\left(\frac{u_t^2}{2}\right), \quad \text{II} = u_{xx} u_t = (u_x u_t)_x - u_x u_{tx}.$$

Then substitution into (*) leads to

$$\rho \left(\frac{u_t^2}{2}\right)_t = T_0 (u_x u_t)_x - T_0 u_x u_{tx}$$

$$\Rightarrow \rho \left(\frac{u_t^2}{2}\right)_t + T_0 u_x u_{tx} = T_0 (u_x u_t)_x$$

$$\Rightarrow \rho \left(\frac{u_t^2}{2}\right)_t + T_0 \left(\frac{u_x^2}{2}\right)_t = T_0 (u_x u_t)_x$$

$$\Rightarrow \frac{\partial}{\partial t}\left[\rho\left(\frac{u_t^2}{2}\right) + T_0\left(\frac{u_x^2}{2}\right)\right] = T_0 (u_x u_t)_x.$$

$$[T_0 (u_x u_t)_x] = \frac{\text{Energy}}{L * t}$$

$\int_0^L dx \rightarrow$ to obtain energy units/time

$$\Rightarrow \underbrace{\frac{d}{dt}\int_0^L \rho\left(\frac{u_t^2}{2}\right)dx}_{\text{Kin. Energy}} + \underbrace{\frac{d}{dt}\int_0^L T_0\left(\frac{u_x^2}{2}\right)dx}_{\text{Potential energy.}} = T_0\int_0^L (u_x u_t)_x\,dx \qquad (2)$$

If

$$\underset{\text{Total Energy}}{E(t)} \equiv \int_0^L \rho\left(\frac{u_t^2}{2}\right)dx + \int_0^L T_0\left(\frac{u_x^2}{2}\right)dx$$

Equ. (2) is equivalent

$$\boxed{\frac{dE}{dt}(t) = T_0 \; u_x u_t(x,t)\Big|_0^L} \qquad (3).$$

\# 4.4.10 What happen to the energy if

a) $u(0,t) = 0, \quad u(L,t) = 0$

$$\text{rhs} = T_0\left[u_x(L,t)\,u_t(L,t) - u_x(0,t)\,u_t(0,t)\right]$$

If $\quad u(0,t) \equiv 0 \Rightarrow u_t(0,t) = 0$

If $\quad u(L,t) \equiv 0 \Rightarrow u_t(L,t) = 0$

$\Rightarrow \quad \frac{dE}{dt}(t) \equiv 0 \Rightarrow E(t) \equiv$ constant (Energy is conserved).

b) This case is similar to (a).

c) If $u(0,t) = 0$, and $[u_x + \gamma u](L,t) = 0$, $\gamma > 0$

$\longrightarrow \gamma > 0$ physically meaningful.

and $$\frac{dE}{dt}(t) = T_0 \left[u_x(L,t)\, u_t(L,t) - u_x(0,t)\, u_t(0,t) \right] \quad (4)$$

then $u_t(0,t) \equiv 0$ and $u_x(L,t) = -\gamma\, u(L,t)$.

Substitution into (4)

$$\frac{dE}{dt}(t) = -T_0\, u_t(L,t)\, \gamma\, u(L,t) = -T_0 \gamma \left(\frac{u^2}{2}(L,t) \right)_t$$

$\Rightarrow$ $\frac{dE}{dt}(t) \leq 0 \Rightarrow$ Energy is decreasing.

d) If $\gamma < 0$ (without physical meaning).

$\frac{dE}{dt} \geq 0 \Rightarrow$ Energy is increasing. !

\# 4.4.1 If $u(x,t) = R(x-ct)$.

$\Rightarrow$ Kin. Energy $= \int_0^L \rho \left(\frac{u_t^2}{2} \right) dx = \int_0^L \rho\, \frac{c^2}{2} R''(x-ct)\, dx$

$\qquad\qquad\qquad\qquad\qquad\qquad \|$ since $c^2 = \frac{T_0}{\rho}$

and Pot. Energy $= \int_0^L T_0 \left(\frac{u_x^2}{2} \right) dx = \int_0^L \frac{T_0}{2} R''(x-ct)\, dx$ ✓

4.4.12 (More general)

<u>Theorem (Tijonov):</u> <u>Uniqueness.</u>

Consider the IBVP.:

$$\begin{cases} \rho(x)\, u_{tt} = \left[T(x)\, u_x\right]_x + F(x,t), & \rho(x)>0,\ T(x)>0 \\ & 0<x<L,\ t>0 \\ \text{I.C's}: u(x,0)=\varphi(x), & u_t(x,0)=\psi(x) \\ \text{B.C's}: u(0,t)=\mu_1(t), & u(L,t)=\mu_2(t) \end{cases}$$

If. ① u, u_{tt}, u_{xx} and u_{xt} are conts on $[0,L]$, $t\geq 0$.

② $\rho(x)$, $T(x)$ continuous on $[0,L]$.

Then the soln. of the above IBVP is unique

<u>Proof.</u>- As usual to prove uniqueness, we assume two solutions u_1 and u_2 and define

$$v(x,t) = [u_1 - u_2](x,t).$$

Therefore, $v(x,t)$ satisfies the corresponding homogeneous problem:

$$\begin{cases} \rho(x)\, v_{tt} = \left[T(x)\, v_x\right]_x \\ v(x,0)=0, \quad v_t(x,0)=0 \\ v(0,t)=0, \quad v(L,t)=0 \end{cases}$$

If we can prove that this homogeneous problem only has a trivial solution

$$v(x,t) \equiv 0 \Rightarrow u_1(x,t) = u_2(x,t).$$

$\Rightarrow$ The original problem only has one solution.

So we will prove next that the homogeneous problem only admits the trivial solution.

Using the expression for the total energy. Assume $T(x) \equiv T_0$ (const.)

$$E(t) = \frac{1}{2} \int_0^L \left[\rho(x) v_t^2 + T_0 \ v_x^2(x,t)\right] dx \qquad (*)$$

and knowing (eq. (3)) that (Assume $T(x) \equiv T_0$ (const.))

$$\frac{dE}{dt}(t) = T_0 \ v_x v_t \Big|_0^L = T_0\left[v_x(L,t)\, \overset{=0}{v_t(L,t)} - v_x(0,t)\, \overset{=0}{v_t(0,t)}\right] = 0$$

$$\Rightarrow \quad E(t) = \text{Constant}.$$

Also, from (*)

$$E(0) = \frac{1}{2} \int_0^L \left[\rho(x) \ \overset{\text{I.C.}\ =0}{v_t^2(x,0)} + T_0 \ \overset{\text{I.C.}\ =0}{v_x^2(x,0)}\right] dx = 0$$

$$\Rightarrow \quad E(t) \equiv 0 \quad \text{(due to continuity)}.$$

$$\Rightarrow \quad E(t) = \frac{1}{2} \int_0^L \left[\overset{>0}{\rho}\, v_t^2(x,t) + \overset{>0}{T_0} \cdot v_x^2(x,t)\right] dx \equiv 0 \Rightarrow \begin{matrix} v_t(x,t) \equiv 0 \text{ and} \\ v_x(x,t) \equiv 0 \end{matrix}$$

$$\Rightarrow v(x,t) \equiv \text{Const.}$$

but $v(x,0) = 0 \xRightarrow{\text{Cont.}} v(x,t) \equiv 0 \Rightarrow \boxed{u_1(x,t) = u_2(x,t)}$