3.6: 1c, 11, 15

1c. Find a basis for the row space, a basis for the column space, and a basis for the nullspace of the matrix

\[
A = \begin{pmatrix}
1 & 3 & -2 & 1 \\
2 & 1 & 3 & 2 \\
3 & 4 & 5 & 6
\end{pmatrix}.
\]

**Answer.** The reduced row echelon form of \(A\) is

\[
B = \begin{pmatrix}
1 & 0 & 0 & -13/20 \\
0 & 1 & 0 & 21/20 \\
0 & 0 & 1 & 3/4
\end{pmatrix}.
\]

Thus the row vectors of \(B\) form a basis for the row space of \(B\), which equals the row space of \(A\). Hence

\[
\langle 1,0,0,-13/20 \rangle, \langle 0,1,0,21/20 \rangle, \text{ and } \langle 0,0,1,3/4 \rangle
\]

form a basis for the row space of \(A\).

The column space of \(A\) has a basis the first three columns of \(A\), since these are the pivot columns in \(B\). Thus

\[
\langle 1,2,3 \rangle^T, \langle 3,1,4 \rangle^T, \text{ and } \langle -2,3,5 \rangle^T
\]

form a basis for the column space of \(A\).

Augmenting \(B\) with the 0 vector, we see that the nullspace of \(A\) is the set of multiples of the vector \(\langle 13/20, -21/40, -3/4, 1 \rangle^T\). Thus

\[
\langle 13/20, -21/40, -3/4, 1 \rangle^T
\]

is a basis for the nullspace of \(A\).

11. Let \(A\) be a \(5 \times 8\) matrix with rank equal to 5 and let \(\mathbf{b}\) be any vector in \(\mathbb{R}^5\). Explain why the system \(A\mathbf{x} = \mathbf{b}\) must have infinitely many solutions.

**Proof.** By Theorem 3.6.6, the dimension of the column space is 5. Thus the column vectors of \(A\) span \(\mathbb{R}^5\). So by Theorem 3.6.3, the linear system \(A\mathbf{x} = \mathbf{b}\) is consistent. Since the dimension of the column space is 5 and \(A\) has 8 columns, we must have that the column vectors of \(A\) are not linearly independent. So by Theorem 3.6.3, the linear system \(A\mathbf{x} = \mathbf{b}\) has more than one solution. If \(\mathbf{x}_1\) and \(\mathbf{x}_2\) are two different solutions, then the vector \(\mathbf{y} = \mathbf{x}_1 - \mathbf{x}_2\) has the property that

\[
A(\mathbf{y}) = tA\mathbf{y} = tA(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = 0
\]

for every scalar \(t\), so \(\mathbf{x}_1 + t\mathbf{y}\) has the property that

\[
A(\mathbf{x}_1 + t\mathbf{y}) = A\mathbf{x}_1 + A(t\mathbf{y}) = \mathbf{b} + \mathbf{0} = \mathbf{b}
\]

for every scalar \(t\). There are infinitely many scalars, and each \(t\) gives a different vector \(\mathbf{x}_1 + t\mathbf{y}\). Therefore, the system \(A\mathbf{x} = \mathbf{b}\) must have infinitely many solutions.

\[\square\]
15. Prove that a linear system $Ax = b$ is consistent if and only if the rank of $(A \mid b)$ equals the rank of $A$.

Proof. Suppose that $Ax = b$ is consistent. Then $b$ is in the column space of $A$ by Theorem 3.6.2, so the dimension of the column space of $A$ equals the dimension of the column space of $(A \mid b)$. By Theorem 3.6.6, the dimension of the column space of $A$ equals the rank of $A$, and the dimension of the column space of $(A \mid b)$ equals the rank of $(A \mid b)$. Therefore, the rank of $(A \mid b)$ equals the rank of $A$.

Now suppose that the rank of $(A \mid b)$ equals the rank of $A$. Then by Theorem 3.6.6, the dimension of the column space of $A$ equals the dimension of the column space of $(A \mid b)$, as in the previous paragraph. Hence a basis for the column space of $A$ is a basis for the column space of $(A \mid b)$. Therefore, $b$ is in the span of the columns of $A$. By Theorem 3.6.2, the linear system $Ax = b$ is consistent. \qed