MS Algebra Exam – February 2018

Answer questions 1–8, Part I. Then answer only two of 9–14, Part II. Partial credit will be given.

Part I Answer all the questions 1–8.

1. Determine the number of elements of order 2 in the dihedral group D_n (symmetries of the regular n-gon).

2. If θ is an automorphism of a group G, and N is a normal subgroup of G, prove that $\theta(N)$ is a normal subgroup of G.

3. Suppose $aba^{-1} = b^i$ in a group G. Prove that $a^k ba^{-k} = b^{i^k}$.

4. Prove that the ring $\mathbb{Z}[x]$ is *not* a principal ideal domain.

5. Prove or disprove the following statement: "The ring $\mathbb{Z}_6$ and the ring $\mathbb{Z}_2 \times \mathbb{Z}_3$ are isomorphic."

6. Let A and B be $n \times n$ matrices over $\mathbb{R}$ and suppose that the eigenvectors of A form a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n\}$ over $\mathbb{R}$. Prove that if each v_j for $j = 1, 2, \ldots, n$ is also an eigenvector of B, then $AB = BA$.

7. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for vector space V. Using only the fact that Span $\mathcal{B} = V$ and $\mathcal{B}$ is linear independent, prove for each $\mathbf{x} \in V$ there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

8. Let V be a vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of linearly independent vectors in V. Now let $S = \{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_2 + \mathbf{v}_3\}$. Prove that S is a linearly independent set.

Part II Answer only two of these:

9. Let

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Compute the eigenprojections P_λ of B. (Hint: If $A = \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & j \\ 0 & 0 & 0 & k \end{bmatrix}$

then $A^{-1} = \begin{bmatrix} 1/a & -b/(ae) & (bf-ce)/(aeh) & (bgh-deh-bfj+cej)/(aehk) \\ 0 & 1/e & -f/(eh) & -(gh-fj)/(ehk) \\ 0 & 0 & 1/h & -j/(hk) \\ 0 & 0 & 0 & 1/k \end{bmatrix}$.)

10. Show $(A^D)^D = A^2 A^D$.

11. Let F be a finite field of characteristic 2. Prove that, for every $a \in F$, the equation $x^2 = a$ has a solution. Show, by example, that the statement is false if F is an infinite field of characteristic 2.

12. Prove that $\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$ is a splitting field. over the rationals.

13. Let G be the subgroup of S_4 generated by (12) and (34). Prove the permutation module for G over field F is not isomorphic to the regular FG-module.

14. Suppose that V is an FG-module with basis $\mathcal{B}$, and let ρ be the representation of G over F defined by

$$\rho : g \to [g]_{\mathcal{B}} \qquad (g \in G).$$

Prove that if σ is a representation of G which is equivalent to ρ, then there is a basis $\mathcal{B}'$ of V such that

$$\sigma : g \to [g]_{\mathcal{B}'} \qquad (g \in G).$$