MS Algebra Exam – January 2018

Answer questions 1–8, Part I. Then answer only two of 9–14, Part II. Partial credit will be given.

Part I Answer all the questions 1–8.

1. Let $K = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (the Klein 4-group). Prove that $\mathrm{Aut}(K)$ (the group of automorphisms of K) is isomorphic to S_3.

2. Prove that a group G is Abelian if and only if the map $\theta : g \mapsto g^{-1}$ is a homomorphism.

3. Let A and B be two subgroups of group G. Prove if $AB = BA$, where $AB = \{ab | a \in A \text{ and } b \in B\}$, then AB is a subgroup of G.

4. In the ring $\mathbb{Z}[z]$, let $I = \langle x, x^2 + 3 \rangle$ (the ideal generated by x and $x^2 + 3$). Prove that I is a maximal ideal.

5. Prove that $\mathbb{Z}_5[x]/(x^2 + 1)$ is isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_5$.

6. Consider the vector space $C_{[-1,1]}$ of continuous functions on $[-1, 1]$. For f and g in $C_{[-1,1]}$ let

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\ dx.$$

   Prove $\langle f, g \rangle$ is an inner product on $C_{[-1,1]}$

7. Let A be an $n \times n$ matrix there $A^T = A^{-1}$. Prove that the columns of A form an orthonormal basis of of $\mathbb{R}^n$.

8. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation give by $T(\mathbf{x}) = A\mathbf{x}$ where A is an $n \times n$ invertible matrix. Prove that T is a one-to-one transformation.

(over)

Part II Answer only two of these:

9. Determine the minimum polynomial for
$$\begin{bmatrix} 5 & 1 & 2 \\ -4 & 0 & -2 \\ -4 & -1 & -1 \end{bmatrix}.$$

10. Show that if $AB = BA$ then $e^{A+B} = e^A e^B$.

11. Let F be a field. If $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ is irreducible in $F[x]$, prove that $c_0 x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n$ is also irreducible in $F[x]$.

12. Let F be a field of order p^k where p is a prime. Prove that $\mathbb{Z}_p$ is a subfield of F, and that F is a splitting field over $\mathbb{Z}_p$.

13. Let $G = C_4 \times C_4$. Prove that there is no irreducible representation σ of G such that $g\sigma = (-1)$ for all elements so of G of order 2 in G.

14. Let G be the cyclic group of order m, say $G = \langle a : a^m = 1 \rangle$. Suppose that $A \in \mathrm{GL}(n, \mathbb{C})$, and define $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ by
$$\rho : a^r \to A^r \qquad (0 \leq r \leq m-1).$$
Show that ρ is a representation of G over $\mathbb{C}$ if and only if $A^m = I$.