MS Algebra Exam – January 2017

Answer questions 1–9. Then answer only one of 10 or 11. Partial credit will be given.

1. If p is an odd prime and $q = 2p + 1$ is also a prime, prove that U_{q^2} (the multiplicative group of units in the ring $\mathbb{Z}/q^2\mathbb{Z}$) is cyclic.

2. Let p be an odd prime and let G be a non-cyclic group of order $2p$. Prove there are p elements of order 2 in G.

3. Prove that the group of rationals under addition is not isomorphic to the multiplicative group of positive rational numbers.

4. Let P be a prime ideal of a commutative ring R. Is $P \times P$ a prime ideal in $R \times R$? Prove your answer.

5. Prove that the principal ideal generated by the element $x + 1$ in $\mathbb{Z}[x]$ is a prime ideal, but not a maximal ideal.

6. Let H be the subset of the vector space $\mathbb{R}^n$, defined by $H = \{\mathbf{x} \in \mathbb{R}^n \big| \, ||\mathbf{x}|| = 1\}$. Prove or disprove the following statement:

$$H \textit{ is a subspace of } \mathbb{R}^n.$$

7. Let $\mathbb{P}^2$ be the vector space of real polynomials of degree 2 or less. Let $\mathbb{M}^{2\times 2}$ be the vector space of 2×2 real valued matrices under matrix *addition.* Define a linear transformation $T : \mathbb{P}^2 \to \mathbb{M}^{2\times 2}$ as

$$T(p) = \begin{pmatrix} p(1) & p(0) \\ p(0) & p(2) \end{pmatrix}$$

   (a) Is T onto? Explain why or why not.
   (b) Is T one-to-one? Explain why or why not.
   (c) Find a basis for $T(\mathbb{P}^2)$, the image of T. Prove it is a basis.

8. Let A and B be $n \times n$ matrices. Prove that

$$\det \begin{pmatrix} A & B \\ 0 & I_n \end{pmatrix} = \det(A)$$

9. Let F be a finite field. Prove that F is an extension of $\mathbb{Z}_p$ (the integers mod p) for some prime number p.

Now answer only one of:

10. Let F be a field, and G a Galois extension of F. If H is an intermediate field, prove one of the following and disprove the other:
    (a) H is a galois extension of F.
    (b) G is a galois extension of H.

11. Let G be a group of order 15. Show that every irreducible character of G has degree 1.