MS Algebra Exam – February 2016

Answer questions $1-9$*. Then answer only one of* $10, 11$*. Partial credit will be given.*

1. Let $\varphi : G \to H$ be a group homomorphism of finite groups that is onto. Show that the order of G is divisible by the order of H.

2. Let H be a subgroup of finite index in a group G. Prove that G has a normal subgroup K of finite index with $K \subseteq H$.

3. Show that there is no subgroup of order 6 in the alternating group A_4.

4. Determine all subrings R of the ring $\mathbb{Z}/(8\mathbb{Z})$.

5. Determine the minimal polynomial for $\sqrt{3} + \sqrt[3]{2}$ over $\mathbb{Q}$.

6. Find all $n \in \mathbb{Z}$ such that
$$n \equiv 1 \mod 4, \text{ and } 2n \equiv 7 \mod 9.$$

7. Find the inverse of the block diagonal matrix $\begin{bmatrix} A & B & 0 \\ 0 & B & C \\ 0 & 0 & D \end{bmatrix}$, given that the square matrices A, B, and D are all invertible.

8. If $\boldsymbol{x}_k = A\boldsymbol{x}_{k-1}$ for $A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$ and $\boldsymbol{x}_0 = \begin{bmatrix} 10.0 \\ 0.0 \end{bmatrix}$, describe the long term behavior of $\boldsymbol{x}_k$?

9. If A and B are in the space of 2×2 matrices with real number entries, then
$$\langle A, B \rangle = \text{trace}(A^T B)$$
is an inner product on this space. Find an orthogonal basis for the subspace
$$H = \left\{ \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

Now answer only one of:

10. Let p be a prime. Find the Galois group of $x^p - 1$ over $\mathbb{Q}$. Prove your result.

11. Let G be a finite group. Show that the kernel and image of a $\mathbb{C}G$ -module homomorphism $\varphi : U \to V$ of $\mathbb{C}G$-modules U, V, are both $\mathbb{C}G$-submodules.