MS Algebra Exam – January 2016

Answer questions $1-9$. *Then answer only one of* $10, 11$. *Partial credit will be given.*

1. Let $G = S_6$, the symmetric group on six elements. Let H be the subgroup of G generated by $(1,2)(3,4)(5,6)$. Find the centralizer $C_G(H)$ i.e. find the subgroup of all elements of G that commute with all the elements of H. (It is enough to give generators for $C_G(H)$.)

2. Let G be a finite group of order n and let $H \leq G$ be a subgroup. Prove that the order of H divides n.

3. Let $G \leq GL(3, \mathbb{F}_3)$, be the group of invertible 3×3 upper triangular matrices over the field with 3 elements (i.e. entries below the diagonal are zero). Find the order of G and show that G is not a simple group i.e. show that G has a proper non-trivial normal subgroup.

4. Prove that $\mathbb{Q}(\sqrt[4]{2}, i)$ is a splitting field over $\mathbb{Q}$.

5. Let R be a commutative ring with identity, and let I be an ideal of R. Prove there is a bijection between the intermediate ideals J such that $I \subseteq J \subseteq R$ and the ideals of the quotient ring R/I. Thus prove that if I is maximal ideal, then R/I is a field.

6. Prove that every finite integral domain is a field.

7. Orthogonally diagonalize the matrix $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix}$, if it is orthogonally diagonalizable. If it isn't explain why not.

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be orthogonal projection onto the plane $2x - z = 0$. Find the standard matrix for T (i.e., the matrix A so that $T(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$). Show that $A^2 = A$.

9. Prove that for A an $m \times n$ matrix and B and $n \times p$ matrix:
$$\text{rank } AB \leq \min\{\text{rank } A, \text{rank } B\}.$$

Now answer only one of:

10. Find the Galois group of $x^4 - 32 \in \mathbb{Q}[x]$.

11. Determine the character table for the symmetric group S_4.