Algebra Masters Exam, January 2015
Answer all questions. Partial credit will be given.

1. Let G be a cyclic group of order $n \in \mathbb{N}$. Show that there is a unique subgroup of order d for every positive divisor d of n.

2. Let $\varphi : G \to H$ be a homomorphism of groups. Let G' and H' denote the commutator subgroups of G and H respectively (these being the smallest subgroup containing all commutators $xyx^{-1}y^{-1}$). Show that $\varphi(G') \subseteq H'$.

3. Prove that $\mathbb{Q}/\mathbb{Z}$, as a quotient of additive groups, is infinite but *not* isomorphic to $\mathbb{Z}$.

4. Let V be the space of continuous functions on the interval [-2,2], with inner product
$$\langle f, g \rangle = \int_{-2}^{2} f(t)g(t)\,dt$$
Find an orthonormal basis (with respect to this inner product) for the subspace spanned by the three functions 1, $2t$, and $\frac{1}{2}t^2$.

5. Prove that if two $n \times n$ matrices over $\mathbb{C}$ are similar then they have the same characteristic polynomial, and hence the same eigenvalues.

6. Let R be a commutative ring. For each ideal $J \subseteq R$ of R, let $\mathrm{rad}(J)$ denote the set
$$\{x \in R : x^n \in J \text{ for some } n \geq 0\}.$$
Let I be an arbitrary ideal of R.
(1) Prove that $\mathrm{rad}(I)$ is an ideal.
(2) Show that $\mathrm{rad}(\mathrm{rad}(I)) = \mathrm{rad}(I)$.
(3) Show that if I is prime, then $\mathrm{rad}(I) = I$.

7. Let M be a maximal ideal in a commutative ring R. Prove that R/M is a field.

8. Let f be the evaluation map from $\mathbb{Z}[x]$ to $\mathbb{Z}$ defined by $f(p) = p(0)$, and let g be the natural map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ (the two-element ring). Let I be the kernel of the composition $g \circ f$. Prove that I is not a principal ideal in $\mathbb{Z}[x]$.

9. Prove that every finite integral domain is a field.

10. Is $x^5 - 6x + 3$ reducible or irreducible over $\mathbb{Q}$? Over $\mathbb{F}_2$? Over $\mathbb{F}_3$? Justify your answers, and if it is reducible, find the factorization into irreducibles..