Algebra Masters Exam, August 2013
Answer all questions. Partial credit will be given.

1. Show that, up to isomorphism, there are only two groups of order 6. (You may use the Sylow theorems.)

2. Prove the First Isomorphism Theorem for groups: If $\varphi : G \to H$ is a group isomorphism, then
   (i) $K = \ker(\varphi)$ is a normal subgroup of G,
   (ii) the image of φ is a subgroup of H, and
   (iii) the quotient group G/K is isomorphic to the image of φ.

3. Choose an explicit Sylow 2-subgroup S of S_4.
   (a) Explicitly describe the elements of your choice of S (as elements of S_4).
   (b) How many distinct conjugates of S are there in S_4?
   (c) Give a set of coset representatives for the cosets of S in S_4.

4. Show that any rational root of a monic polynomial over $\mathbb{Z}$ is an integer.

5. Prove that for every pair of proper comaximal ideals $I, J \subseteq R$ of a commutative ring R with $1 \neq 0$, we have $IJ = I \cap J$. Find an example showing that this is not true if we drop the comaximality hypothesis.

6. Let m, n be relatively prime positive integers. Show that $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ (as rings).

7. What is the dimension, over $\mathbb{R}$, of the vector space of $n \times n$ real symmetric matrices of trace zero? Justify your answer.

8. Let F be a field and let α be algebraic over F. Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha^2) = F(\alpha)$.

9. Let F be a field, and let K/F be a finite extension. Prove that K/F is a splitting extension if and only if every irreducible polynomial $p(x) \in F[x]$ with a root in K splits completely in K.

10. Prove that the order of a finite field is a power of a unique prime.