# MASTER'S EXAM, ALGEBRA, FEBRUARY 2013

1. If $M$ is a $4 \times 4$ complex matrix with distinct eigenvalues $e_1, e_2, e_3, e_4$, find the coefficients of the characteristic polynomial of $M$.

2. Let $A$ and $B$ be matrices such that $AB$ is defined. Prove that $\operatorname{rank} AB \leq \min(\operatorname{rank} A, \operatorname{rank} B)$.

3. Show that if $A$ is a real symmetric matrix with all eigenvalues nonnegative, then there is a real symmetric matrix $S$ such that $S^2 = A$.

4. Let $J, H$ be subgroups of finite index in a group $G$. Show that the intersection $H \cap J$ also has finite index.

5. Show that a group of order 45 has a normal subgroup of order 9.

6. Prove that there is a homomorphism from the alternating group $A_4$ onto the cyclic group of order 3.

7. Let $F$ be a field and let $a, b \in F[x]$ with $b \neq 0$. Prove that there are unique polynomials $q, r \in F[x]$ such that $a = bq + r$, with $r = 0$ or $\deg(r) < \deg(b)$.

8. Let $I$ be the principal ideal in $\mathbb{R}[x]$ generated by the polynomial $x^2 - 3x$. Let $\theta : \mathbb{R}[x] \to \mathbb{R} \times \mathbb{R}$ be defined by $\theta(f(x)) = (f(0), f(3))$ and prove that $\mathbb{R}[x]/I \cong \mathbb{R} \times \mathbb{R}$.

9. (a) Construct a field with 9 elements.
   (b) Find a generator for the multiplicative group of this field.

10. Let $\alpha$ be the real cube root of 2. Let $\beta = 1 + \alpha^2$. Find the minimal polynomial of $\beta$ over $\mathbb{Q}$.