MASTER'S EXAM, ALGEBRA, FEBRUARY 2012

1. Show that the determinant of an orthogonal matrix must have absolute value 1.

2. Let A be a nilpotent $n \times n$ real matrix. (The matrix A is said to be nilpotent if $A^k = 0$ for some positive integer k.) Show that the only possible eigenvalue of A is 0.

3. Let $\mathcal{P}_3$ be the set of polynomials with real coefficients of degree no more than 3. Show that $\{1, x+1, x^2 - x + 1, 1 - x^3\}$ forms a basis for $\mathcal{P}_3$. Compute the matrix for the linear map given by differentiation of polynomials with respect to that basis. Compute a basis for the nullspace of the matrix.

4. Let G be a group, H a subgroup of G, and $a \in G$. Let m be the order of a, and let n be the smallest positive integer such that $a^n \in H$. Prove that $n|m$.

5. Assume that every element of a group G has order ≤ 2. Prove that G is abelian.

6. Show that any group of order 14 has a normal subgroup of order 7.

7. Let R be a finite commutative ring with $1 \neq 0$. Prove that if R has no zero divisors, then R is a field.

8. Let F be a field. Show that the ring $F[x]$ of polynomials in one variable over F is a principal ideal domain.

9. Prove that if E is a Galois extension of $\mathbb{Q}$ of odd degree, then $E \subset \mathbb{R}$.

10. Show that if E/F is a finite extension of fields, then any subring R of E containing F is itself a field.