MASTER'S EXAM, ALGEBRA, JANUARY 2012

1. Let S be a subset of the inner product space V. Show that the set of vectors orthogonal to S forms a subspace of V.

2. Show that no set of 3 vectors can span $\mathbb{R}^4$.

3. Let A be a square matrix with real entries. Show that two (non-zero) eigenvectors of A which correspond to different eigenvalues must be linearly independent.

4. Let $\mathbb{Z}$ denote the additive group of integers, and let $G = \mathbb{Z} \times \mathbb{Z}$ be the direct product of $\mathbb{Z}$ with itself. Let H be the subgroup of G defined by

$$H = \{(2m+n, 3n) \big| m, n \in \mathbb{Z}\}.$$

   Prove that $G/H \equiv \mathbb{Z}/6\mathbb{Z}$.

5. Let G be a group, and suppose that M and N are both normal subgroups of G. Suppose that $M \cap N = \{e\}$, where e is the identity of G. Prove that for any $m \in M$ and $n \in N$, $mn = nm$.

6. Let G be the alternating group A_4. Find a Sylow p-subgroup of G for each prime p dividing the order of G.

7. Recall that $\mathbb{R}[x]$ denotes the ring of polynomials in x with real coefficients. Let I be the principal ideal generated by the polynomial x^2-1. Let $\psi : \mathbb{R}[x] \to \mathbb{R} \times \mathbb{R}$ be defined by $\psi(p(x)) = (p(1), p(-1))$ and prove that $\mathbb{R}[x]/I \cong \mathbb{R} \times \mathbb{R}$.

8. Recall that a polynomial is **monic** if the leading coefficient is 1. Find (with proof) the number of monic irreducible polynomials in $\mathbb{F}_3[x]$ of degree 6. Here $\mathbb{F}_3$ is the field with three elements.

9. Calculate the seventy-second cyclotomic polynomial $\Phi_{72}(x)$.

10. Given that the discriminant of the cubic polynomial $ax^3 + bx^2 + cx + d$ is $b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$, calculate the Galois group over the rationals of the following polynomials:

    (a) $x^3 + 2x^2 + 3x + 4$

    (b) $x^3 - 2x^2 - x + 1$