MASTER'S EXAM, ALGEBRA, AUGUST 2011

1. A matrix A is said to be idempotent if $A^2 = A$. Show that every finite, real, symmetric idempotent matrix represents a linear transformation which is an orthogonal projection and identify the subspace onto which it is projecting.

2. Prove that any set of 3 vectors in $\mathbb{R}^2$ is linearly dependent.

3. State and prove the Rank Plus Nullity theorem for $\mathbb{R}^n$.

4. Let G be a group. Show that if each element of G has order 2, then G is abelian.

5. If G is a group, show that if H is a normal subgroup of G, then the multiplication of right cosets $Hg_1Hg_2 = Hg_1g_2$ is well defined.

   Find an example of a group G with a subgroup H for which this multiplication would not be well defined.

6. Let G be a group of order 245. Show that G has a normal subgroup of order 49.

7. Find an ideal in $\mathbb{Z} \times \mathbb{Z}$ that is prime but not maximal.

8. Let R be a commutative ring with $1 \neq 0$. Suppose I and J are ideals of R such that $I + J = R$. Prove that

$$R/(I \cap J) \cong R/I \oplus R/J.$$

9. Prove that the Frobenius map on a field of finite characteristic is always injective and give an example of a field of finite characteristic for which it is not surjective.

10. Prove that the order of a finite field is a power of a prime.