MASTER'S EXAM, ALGEBRA, FEBRUARY 2011

1. If the columns of an orthogonal matrix are permuted, prove that the result is still an orthogonal matrix.

2. Are the matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ similar? Explain.

3. Let U be a finite-dimensional vector space, and let V and W be subspaces of U. Give a formula relating the dimensions of V, W, $V+W$, and $V \cap W$. Prove that your formula is correct. (Note: You may use without proof the vector space analogues of the homomorphism theorems from group theory.)

4. Show that there is no simple group of order 148.

5. Let H be a subgroup of finite index of an infinite group G. Prove that G has a normal subgroup K of finite index in G with $K \subset H$.

6. Determine the last 3 digits of the number 13^{2011}. Explain your method.

7. (a) Let A be a commutative ring with 1. An element $a \in A$ is said to be nilpotent if $a^n = 0$ for some positive integer n. Prove that the nilpotent elements of A form an ideal in A.

   (b) Does the result of part (a) still hold if the hypothesis of commutativity is dropped? Prove or disprove.

8. Let R be a commutative ring with 1. Prove that the principal ideal (x) in the polynomial ring $R[x]$ is a maximal ideal if and only if R is a field.

9. Prove that the Galois group of the splitting field of $x^4 - 2$ over $\mathbb{Q}$ has order 8 and contains an element of order 4.

10. Let F be a field with 81 elements. Does the polynomial x^2+1 have a root in this field? (The polynomial should be considered as having coefficients in $\mathbb{Z}/3\mathbb{Z}$.)