MASTER'S EXAM, ALGEBRA, JANUARY 2011

1. Let A be an invertible $n \times n$ matrix with integer entries. Prove that A^{-1} has integer entries if and only if $\det A = \pm 1$.

2. (a) Find a 3×3 matrix A that has eigenvalues 0, 1, -1 with corresponding eigenvectors $(0, 1, -1)^T, (1, -1, 1)^T, (0, 1, 1)^T$.

   (b) Is this answer unique? Explain.

3. Prove that if the vectors x, y, z in a vector space V are linearly independent, then so are the vectors $x + y, y + z, x + z$.

4. Let G be a group, and let H be a subgroup of G of finite index n. Let e be the identity of G.

   (a) Show that if H is normal in G, then $x^n \in H$ for all $x \in G$.

   (b) Is the statement in the first part true if the hypothesis of normality is dropped? Prove or disprove.

5. How many elements of order 5 are contained in a group of order 20? Justify your answer.

6. If $\gcd(m, n) = d$, prove that the system

$$x \equiv a \pmod{m}$$

$$x \equiv b \pmod{n}$$

   has a solution if and only if $a \equiv b \pmod{d}$. If s, t are solutions of this system, prove that $s \equiv t \pmod{r}$, where r is the least common multiple of m and n.

7. (a) Determine all ideals of the ring $R = \mathbb{Z}[x]/(2, x^3 + 1)$.

   (b) Determine the number of elements of R.

   (c) Write R as a direct product of fields if possible.

8. Determine the number of monic irreducible cubic polynomials in $F_p[x]$, where F_p denotes the field with p elements.

9. Factor each of the following into irreducible factors in $\mathbb{Q}[x]$. Justify your answer.

   (a) $x^4 + 4$

   (b) $x^4 - 4x^3 + 6$

   (c) $x^3 + x + 1$

10. Give an example of an infinite field of characteristic 5.