MASTER'S EXAM, ALGEBRA, AUGUST 2010

1. Prove that a finite orthogonal set of non-zero vectors is linearly independent.

2. Are the matrices $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 9 \\ 0 & 0 & 6 \end{bmatrix}$ and $\begin{bmatrix} 6 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ similar? Explain.

3. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ in exactly one way.

4. Prove that there is no simple group of order 56.

5. Let A_4 be the alternating group (the set of even permutations) on 4 elements. Prove that A_4 has no subgroup of order 6.

6. Determine the last 3 digits of the number 9^{2005}. Explain your method.

7. A polynomial in $\mathbb{Z}[x]$ is said to be primitive if there is no prime which divides all the coefficients. Prove that the product of primitive polynomials is primitive.

8. The ring of formal power series $R = F[[t]]$ with coefficients in a field F is the set of all series $a_0 + a_1t + \ldots$, $a_i \in F$, added and multiplied in the manner familiar from analysis.
   (a) Prove that the series $a_0 + a_1t + \ldots$ has a multiplicative inverse in R if and only if $a_0 \neq 0$.
   (b) Determine all ideals in R.
   (c) Is R a unique factorization domain?

9. Let ζ be a primitive 5th root of unity. Let $K = \mathbb{Q}(\zeta)$.
   (a) Find the degree of $K/\mathbb{Q}$.
   (b) Find the Galois group of $K/\mathbb{Q}$.
   (c) Find all intermediate fields between K and $\mathbb{Q}$ and indicate which ones are Galois over $\mathbb{Q}$.

10. Let F be a field with 64 elements. Does the polynomial $x^3 - x + 1$ have a root in this field? Justify your answer. (The polynomial should be considered as having coefficients in $\mathbb{Z}/2\mathbb{Z}$.)