Master's Analysis Exam – February 2018

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY AND TRY TO KEEP AT LEAST AN INCH MARGIN ALL AROUND.

1. Show that the finite union of countable sets is countable.
2. Show that if $a, b \in \mathbb{R}$ with $a < b$, then there exists a rational number q with $q \in (a, b)$.
3. If (a_n) is a convergent sequence, prove that (a_n) is a Cauchy sequence.
4. For a nonempty subset A of $\mathbb{R}^n$, show that if every sequence $(a_n) \subset A$ has a convergent subsequence which converges to a point in A (i.e. A is compact), then A is closed.
5. If $f : [0, 1] \to \mathbb{R}$ is twice differentiable at every point with $f''(x) \leq M$, and if $f(0) = f'(0) = 0$, show that $f(x) \leq Mx^2/2$ for all $x \in [0, 1]$.
6. Let $f(x, y)$ have continuous second order derivatives in a bounded and open domain $\Omega \subset \mathbb{R}^2$. If $f_{xx} \geq 1$ everywhere in Ω, and if f achieves a maximum at $(x_0, y_0) \in \Omega$, show that (x_0, y_0) is on the boundary of Ω.
7. Show that the vector field $\dfrac{\mathbf{r}}{|\mathbf{r}|^p}$ in $\mathbb{R}^2$ (where $\mathbf{r} = \langle x, y \rangle$ and $p \in \mathbb{R}$) is conservative.
8. Use Stokes' theorem for the vector field $F = \langle 0, 0, x^2/2 \rangle$ in order to show (or show directly) that
$$\int_{S^+} \langle 0, -x, 0 \rangle \cdot \nu = 0,$$
where S^+ is the upper half unit sphere (i.e. $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}$), and ν is the outward unit normal to S^+.
9. (L'Hopital's Rule) If $f, g : \mathbb{C} \to \mathbb{C}$ are analytic and $f(0) = g(0) = 0$, and $\lim_{z\to 0} f'/g' = c \in \mathbb{C}$, show that
$$\lim_{z\to 0} \frac{f}{g} = c.$$
Hint: consider the power series representations at the origin of f and g.
10. Prove that the only bounded, entire functions are constants.