Master's Analysis Exam – January 2017

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY.

1. A real number of the form $m/2^n$, where m and n are integers, is called a dyadic rational. Prove that the set of dyadic rationals is dense in $\mathbb{R}$.

2. Prove that if $\{a_n\}$ is a Cauchy sequence in $\mathbb{R}$, then $\{a_n\}$ is bounded.

3. For a metric space X with metric d and an nonempty subset $A \subset X$, prove the function $x \to \operatorname{dist}(A, x)$ is continuous at each $x \in X$. [Recall that $\operatorname{dist}(A, x) = \inf\{d(y, x) : y \in A\}$.]

4. Show that if f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$ for some positive a, then f is uniformly continuous on $[0, \infty)$.

5. Let $f : [a, b]$ be continuous on $[a, b]$ and differentiable in (a, b). Show that if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and is equal to A.

6. Let $c : \mathbb{R} \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ and write $c(t) = (x(t), y(t), z(t))$ and $h(t) = f(c(t))$. Prove that if $c(t)$ is differentiable at t_0, f is differentiable at $c(t_0)$, and the partial derivatives of f are continuous at $c(t_0)$, then $h(t)$ is differentiable at t_0 and

$$\frac{dh}{dt}(t_0) = \frac{\partial f}{\partial x}(c(t_0))\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(c(t_0))\frac{dy}{dt}(t_0) + \frac{\partial f}{\partial z}(c(t_0))\frac{dz}{dt}(t_0).$$

7. Let C be the unit circle centered at the origin in $\mathbb{R}^2$. Evaluate the line integral

$$\int_C (e^x - y^3)dx + (x^3 + y^3)dy.$$

8. If f and g are continuous on $[a, b]$ and $g(x) > 0$ for all $x \in [a, b]$ show that there exists a $c \in [a, b]$ such that $\int_a^b fg = f(c) \int_a^b g$.

9. Suppose that a function and its conjugate are both analytic in a given domain D. Show that the function must be constant.

10. Prove that if f is analytic inside and on a circle of radius r and center z_0 in $\mathbb{C}$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\ d\theta.$$