Master's Analysis Exam – January 2016

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Give an $\epsilon - \delta$ proof that if real sequences $\{a_n\}$ and $\{b_n\}$ converge, then so does their sum $\{a_n + b_n\}$.
2. Prove that if $\lim (a_n/n) = L > 0$ then $\lim a_n = +\infty$.
3. Show that if $f : A \to \mathbb{R}$ is continuous on $A \subseteq \mathbb{R}$ and if $n \in \mathbb{N}$, then the function f^n defined by $f^n(x) = (f(x))^n$ for $x \in A$, is continuous on A.
4. Let M be a metric space with metric d. Prove that if C is a complete subset of M, then C is closed.
5. Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous, f' exists on $(0, \infty)$, $f(0) = 0$, and f' is increasing on $(0, \infty)$. Prove that $g(x) = f(x)/x$ is increasing on $(0, \infty)$.
6. Suppose that the mapping $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and that the derivative matrix $\mathbf{DF}(\mathbf{x})$ at each point $\mathbf{x} \in \mathbb{R}^n$ has all its entries equal to 0. Prove that the mapping $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is constant; that is, there is some point $\mathbf{c}$ in $\mathbb{R}^m$ such that
$$\mathbf{F}(\mathbf{x}) = \mathbf{c} \qquad \text{for every } \mathbf{x} \text{ in } \mathbb{R}^n.$$
7. Consider the integral
$$\int_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$
Evaluate it when:
   (a) C is the circle $x^2 + y^2 = 1$, traversed counter clockwise.
   (b) C is the ellipse $x^2 + \frac{y^2}{4} = 1$, traversed counter clockwise.
8. Prove that $\int_{-1}^{1} 1/\sqrt{1 - x^2}\, dx$ exists.
9. For γ a circle of radius 3 and center at 0 in $\mathbb{C}$, compute
$$\int_\gamma \frac{e^z + z}{z - 2}\, dz.$$
10. Show that the polynomial $z^6 + 4z^2 - 1$ has exactly two zeros in the disk $z < 1$.