Master's Analysis Exam – September 2015

4 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. For a bounded sequence $\{a_n\}$ of real numbers, let L be the set of limits of convergent subsequences of $\{a_n\}$. Prove that if $\sup L = \inf L$, then $\{a_n\}$ converges.

2. Let $\{b_n\}$ be a bounded sequence of nonnegative numbers and let r be any number such that $0 \le r < 1$. Define
$$s_n = b_1 r + b_2 r^2 + \cdots + b_n r^n \quad \text{for every } n \ge 1.$$
Prove the sequence $\{s_n\}$ converges.

3. For a compact interval C of $\mathbb{R}$, let $f : C \to \mathbb{R}$ be continuously differentiable. Prove that f is Lipschitz on C.

4. Prove that if K_1 and K_2 are sequentially compact subsets of $\mathbb{R}^n$ then there exists points $\mathbf{x}_1 \in K_1$ and $\mathbf{x}_2 \in K_2$ such that if $\mathbf{z}_1 \in K_1$ and $\mathbf{z}_2 \in K_2$, then $\|\mathbf{z}_1 - \mathbf{z}_2\| \ge \|\mathbf{x}_1 - \mathbf{x}_2\|$.

5. Let $f : [a, b] \to [a, b]$ be differentiable. Show that f is a contraction of $[a, b]$ if and only if there exists $k < 1$ with $|f'(x)| \le k$ for all $x \in (a, b)$.

6. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose partial derivatives exist at $(0, 0)$, but for which f is not differentiable at $(0, 0)$.

7. Evaluate
$$\iint_S (x - y)^2 \sin^2(x + y)\, dx\, dy$$
where S is the parallelogram with vertices $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$, $(0, \pi)$.

8. Let D be a region in $\mathbb{R}^2$ for which Green's Theorem applies, and let C be the boundary curve of D. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable. Prove that if $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$ on D, then $\int_C \partial f/\partial x\, dx - \partial f/\partial x\, dy = 0$.

9. Suppose $f : \mathbb{C} \to \mathbb{C}$ is complex analytic. Prove that $g(z) = \overline{f(\overline{z})}$ is complex analytic.

10. Evaluate the integral
$$\int_0^\infty \frac{1}{(1 + x^2)x^{1/2}} dx.$$