Master's Analysis Exam – January, 2015

3 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Prove that a bounded sequence of real numbers has a convergent subsequence.
2. If $\{a_n\}$ is a Cauchy sequence of real numbers, and $f : \mathbb{R} \to \mathbb{R}$ is continuous, it is true that the sequence $f(a_n)$ is Cauchy?
3. Let A, B be nonempty subsets of $\mathbb{R}^n$. Prove that $A \times B$ (in $\mathbb{R}^{2n}$) is an open set if both A and B are open.
4. Suppose X and Y are metric spaces and $f_n : X \to Y$. Prove that if $f_n \to f$ uniformly and each f_n is continuous, then f is continuous.
5. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ have continuous first order partial derivatives and suppose that U is open. Prove that for any two points (x, y) and (a, b) in U sufficiently close together,
$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \epsilon_1(x,y)(x-a) + \epsilon_2(x,y)(y-b)$$
and that $\epsilon_1, \epsilon_2 \to 0$ as $(x, y) \to (a, b)$.
6. If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$, where U is open, and f has a local minimum at $(a, b) \in U$. Prove that $\nabla f(a, b) = 0$.
7. Use Green's theorem to evaluate the line integral
$$\int_\Gamma xy\,dx + y^2 dy,$$
where Γ is the perimeter of the rectangle $[0, 4] \times [0, 1]$ oriented in the positive direction.
8. An integral in x, y, z space was converted to $\iiint_E \rho^3 \sin\phi \, dV$ using spherical coordinates. What was the original function $f(x, y, z)$ being integrated?
9. Let f be analytic in a region Ω and $a \in \Omega$. If $f^{(\nu)}(a) = 0$ for $\nu = 0, 1, 2, 3, \cdots$, prove that $f(z) = 0$ for all $z \in \Omega$.
10. Find the exact value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.