Master's Analysis Exam – September, 2013

3 Hours. No notes, textbooks, or calculator

If asked to show something, you must derive it from simpler results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Prove that a bounded monotone sequence $\{x_n\}$ of real numbers converges.

2. Suppose that f is a uniformly continuous mapping of a metric space X into a metric space Y. Prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X.

3. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

4. Prove or disprove: for $A \subseteq \mathbb{R}^n$, if $F : A \to \mathbb{R}^m$ is continuous and A is open, then $F(A)$ is open.

5. Let f be a real-valued function defined on $[a, b]$. If f has a local maximum at a point $c \in (a, b)$ and if $f'(c)$ exists, prove that $f'(c) = 0$.

6. State the Mean Value Theorem for a continuously differentiable function f on $\mathbb{R}^n$.

7. Let D be a (compact) Jordan domain in $\mathbb{R}^k$. If a sequence of continuous functions $\{f_n\}$ on D converges uniformly to f, prove that (a) f is integrable on D, and (b) that

$$\lim_{n\to\infty} \int_D f_n = \int_D f.$$

8. Let f be a continuously differentiable function on a simply connected open subset U of $\mathbb{R}^n$. Prove for any rectifiable simple closed curve γ in U that

$$\int_\gamma \langle \nabla f, d\gamma \rangle = 0,$$

where $\langle x, y \rangle$ is the standard inner product in $\mathbb{R}^n$.

9. Find a linear transformation of $\mathbb{C}$ which carries $|z| = 1$ and $|z - 1/4| = 1/4$ into concentric circles.

10. Find the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$