MASTER'S ANALYSIS EXAM – FEBRUARY 2013

3 hours. No notes, textbooks or calculator.

If asked to show something, you must derive it from **simpler** results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Find the values of x in $\mathbb{R}$ for which the series
$$\sum_{n=2}^{\infty} \frac{x^n}{2^{-n} + n(\log n)}$$
is
   (a) convergent
   (b) absolutely convergent

2. Suppose that (a_n) is a sequence of real numbers converging to ℓ. Show that
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = \ell.$$

3. Let $f : [0,\infty) \to \mathbb{R}$ be continuous. Suppose that f is differentiable on $(1,\infty)$ and f' is bounded on $(1,\infty)$. Show that f is uniformly continuous on $[0,\infty)$.

4. Assuming only the 'sup axiom' (completeness axiom) for bounded nonempty sets in $\mathbb{R}$, show that any bounded sequence in $\mathbb{R}^h$ has a convergent subsequence.

5. Give an example of a function $f(x,y)$ such that $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ both exist, but $f(x,y)$ is not continuous at $(0,0)$.

6. Let
$$f(x) = \begin{cases} |x|^p \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$
For which positive values of p is f differentiable at 0?

7. Evaluate $\displaystyle\iint_{\Omega} xy\,dxdy$, where Ω is the first-quadrant region bounded by the curves
$$x^2 + y^2 = 16\ ,\ x^2 + y^2 = 25,$$
$$x^2 - y^2 = 9\ \ ,\ x^2 - y^2 = 6.$$

8. State Fubini's theorem for $\mathbb{R}^m \times \mathbb{R}^n$. Integrate the function $(\sin x)e^{-xy}$ over the set $(0, a) \times (0, \infty)$. Show that

$$\int_0^a \frac{\sin x}{x} = \frac{\pi}{2} - \cos a \int_0^\infty \frac{e^{-ay}}{1+y^2}\, dy - \sin a \int_0^\infty \frac{y e^{-ay}}{1+y^2}\, dy.$$

9. Let $f : \mathbb{C} \to \mathbb{C}$ have a derivative at the point $z = a + ib$. Writing $f = u + iv$, show that u and v satisfy the Cauchy-Riemann equations at $(x, y) = (a, b)$.

   If f is analytic in an open set U, prove that u is harmonic in U.

10. Evaluate

$$\int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \sin\theta}.$$