

MASTER'S ANALYSIS EXAM – JANUARY 2013

3 hours. No notes, textbooks or calculator.

If asked to show something, you must derive it from **simpler** results. For instance, you may not prove the intermediate value theorem by quoting a theorem about the continuous image of a connected metric space.

1. Let A be the set of algebraic numbers, i.e. real or complex solutions of equations

$$a_n x^n + \cdots + a_1 x + a_0 = 0$$

with integers a_j and $a_n \neq 0$. Show that A is countable.

2. Let a_1, a_2, \dots be complex numbers such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1.$$

Show that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

3. Let X and Y be metric spaces and suppose that $f : X \rightarrow Y$ is continuous. Show that if E is a compact subset of X , then $f(E)$ is compact.

4. Suppose that f is a real continuous function on $[a, b]$. Prove the intermediate value theorem: if y lies between $f(a)$ and $f(b)$, there is c in $[a, b]$ such that $f(c) = y$.

5. Suppose that the mapping $F : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ has the property that each of its partial derivatives $\frac{\partial F}{\partial x_j}$ exist and are bounded on A . If $x \in A$ is an interior point of A , Prove that F is continuous in x .

6. Using $\int_0^x i e^{it} dt$, show that

$$|e^{ix} - 1| \leq |x| \quad (x \in \mathbb{R}).$$

Arguing similarly, obtain

$$|e^{ix} - 1 - ix| \leq \frac{x^2}{2} \quad (x \in \mathbb{R}).$$

7. Define the upper and lower Riemann sums $U(P, f)$ and $L(P, f)$ for a real bounded function f on $[a, b]$ and a partition P of $[a, b]$. State a criterion for Riemann integrability of f in terms of the quantities $U(P, f)$, $L(P, f)$.

Suppose f is Riemann integrable on $I = [a, b]$ and $g : f(I) \rightarrow \mathbb{R}$ satisfies the condition

$$|g(Y) - g(X)| \leq K |Y - X|$$

for X, Y in $f(I)$. Show that $g \circ f$ is Riemann integrable on $[a, b]$.

8. Suppose that the series $f(x) = \sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly for $a \leq x \leq b$.

Show that

$$\int_a^b f(x) dx = \sum_{n=1}^{\infty} \frac{a_n (\cos na - \cos nb)}{n}.$$

9. Suppose that $f : U \rightarrow \mathbb{C}$ is continuous, where U is open and $F : U \rightarrow \mathbb{C}$ satisfies $F' = f$. Show that $\int_C f(z) dz = 0$ for any piecewise smooth closed curve C in U .

10. Find the Laurent series of the function $\frac{1}{z}$ in the region $|z - 1| > 1$.