Masters's Exam, Analysis, January 2012

1. Discuss the convergence or divergence of each series $\sum a_n$ where

   (a) $a_n = \dfrac{(-1)^n}{(n+1)^{7/3}}$

   (b) $a_n = \dfrac{n^s}{n^t+1}, \quad 0 < s < t$

   (c) $a_n = \dfrac{n!}{2^{n+4}}$

   (d) $a_n = \dfrac{\ln(\ln(n))}{n\ln(n)}$

2. Let $f_n \colon \mathbb{R} \to \mathbb{R}$ and $g_n \colon \mathbb{R} \to \mathbb{R}$ be defined for $n \in \mathbb{N}$. Assume, for each n, that both f_n and g_n are bounded. Assume the sequences (f_n) and (g_n) converge uniformly to f and g, respectively. Show that $(f_n g_n)$ converges uniformly to fg.

3. Let $f \colon \mathbb{R} \to \mathbb{R}$ is continuous and let (x_n) be a sequence in $\mathbb{R}$ with $\lim_{n \to x_n} = A$. By directly using the definition of continuity and the definition of the limit of a sequence (and without using theorems about limits) prove that

$$\lim_{n\to\infty} f(x_n) = f(A).$$

4. Suppose A is a compact set in $\mathbb{R}^m$ and $f \colon A \to \mathbb{R}^n$ is continuous. Show that f is bounded.

5. Let

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   Show that $f_x(0,0)$ exists but f_x is not continuous at the origin.

6. Find and classify the critical points of $f(x,y) = xy^2(4-x-y)$. Then find the maximum value of f on the closed triangular region in the xy-plane with vertices $(0,0), (0,6)$ and $(6,0)$.

7. Compute the integral

$$\iint_R (x^2+y^2)^3 \, dx \, dy$$

   where R is the region in the first quadrant that is bounded by the hyperbolas $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, $x^2 - y^2 = 4$.

8. Suppose f is integrable over $[a,b]$. Let $c \in (a,b)$. By using the definition of Riemann-Darboux integrability rather than by quoting theorems about integrals, show that f is integrable over $[a,c]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

9. Find the Laurent series for the function $f(z) = \dfrac{1}{z(2-z)}$

   (a) in the region $|z| > 2$

   (b) in the region $|z-1| < 1$.

10. Compute the integral $\displaystyle\int_0^\infty \frac{\cos(x)}{(x^2+1)^3}\,dx$.