PH. D. QUALIFYING EXAM JANUARY 2018 - ALGEBRA

Answer all the questions

1. Let p, q be distinct primes. Prove that any group of order pq is a semi-direct product of non-trivial groups.

2. Let p be an odd prime. Describe four non-isomorphic groups of order p^3, and prove they are non-isomorphic.

3. Let $F = \langle x, y \rangle$ be a free group of rank 2. Write down a finite set of generators for a subgroup of F that has index 2.

4. Let M be an $n \times n$ matrix over $\mathbb{C}$ with finite order. Prove that M is diagonalizable.

5. Let $C_3 = \langle t \rangle$ denote the cyclic group of order 3 and let $V = \mathbb{C}C_3$ be its complex group algebra. The element $e_1 = (1 + t + t^2)/3$ is an idempotent (i.e. $e_1^2 = e_1$). Find two more idempotents e_2, e_3 such that e_1, e_2, e_3 are orthogonal (i.e. $e_i e_j = 0$ for $i \neq j$). Show that e_1, e_2, e_3 is a basis for V.

6. Find the Galois group of the polynomial $x^8 - 2 \in \mathbb{Q}[x]$.

7. Let F be a field, and let $f(x) \in F[x]$ be an irreducible polynomial. Assume that there is an extension field E/F containing a root α of $f(x)$. Further suppose $f(\alpha^2) = 0$. Prove that f splits completely over E.

8. Let R be a commutative ring with 1. Prove that every prime ideal Q contains a prime ideal P such that P contains no proper prime ideal.

9. Gauss' Lemma says that if R is a UFD with field of fractions F, and $f(x) \in R[x]$ is a monic polynomial, then $f(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. Prove this lemma.

10. Prove that if R is a finite commutative ring with 1, then every prime ideal is maximal.

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