Algebra Ph.D. Qualifying Exam

January 2017

Answer all 10 questions. Your judgment as to which theorems are appropriate to use for each problem is part of what is being tested, so be sure not to use theorems which make the problems trivial.

1. Prove that a group G of order 150 cannot be simple (by considering the action of G on $\mathrm{Syl}_5(G)$ by conjugation or otherwise).

2. What is the abelianization (expressed as a product of cyclic groups) of the group with presentation
$$\langle a, b, c, d \mid (ab)^2(cd)^4, (ac)^4(bd)^8, (ad)^8(bc)^2\rangle.$$

3. Let p be a prime and G a group with $|G| = p^n$. Let F denote the intersection of all the maximal subgroups of G. Show that F is normal in G, and that G/F is abelian.

4. Let R be a non-zero ring with unity. Show that $R \otimes_{\mathbb{Z}} (\mathbb{Z}[x]) \cong R[x]$.

5. Let R be an integral domain and let M be a free R-module of rank $n \in \mathbb{N}$. Show that any $n+1$ elements of M are R-linearly dependent.

6. Let $\mathcal{P}$ be a finite set of odd prime numbers. Prove that there is an irreducible quadratic polynomial $f \in \mathbb{Z}[x]$ such that $f(x)$ has a root mod p for every prime $p \in \mathcal{P}$.

7. Let R be a commutative ring with $1 \neq 0$. Assume that I is an ideal of R such that $1 + a$ is a unit in R for all $a \in I$. Prove that I is contained in every maximal ideal of R.

8. Prove that $\mathbb{Q}(\sqrt{5})$ is the unique quadratic subfield of $\mathbb{Q}(\zeta_5)$, where $\zeta_5 = e^{2\pi i/5} \in \mathbb{C}$.

9. Recall that a primitive element of a field extension L/K is an element $\alpha \in L$ such that $L = K(\alpha)$.

   Let p be a prime number. Determine the number of primitive elements of the field extension $\mathbb{F}_{p^{24}}/\mathbb{F}_{p^2}$.

10. Let R be a finite, nonzero commutative ring (possibly without identity). Prove that if R has no nonzero zero-divisors, then R is a field.