Algebra PhD Qualifying Exam

May 2016

Answer all questions. Partial credit will be given.

1. Let G be a group of order p^n, where p is prime. Show that, for each $0 \leq k \leq n$, the group G has a normal subgroup of order p^k.

2. Construct a non-abelian group G of order 21 and determine the sizes of the conjugacy classes of G.

3. Show that there is no simple group of order $3393 = 3^3 \cdot 13 \cdot 29$.

4. $x^4 + x^2 - 6$. Let $f(x) = x^3 + \frac{2}{27} \in \mathbb{Q}[x]$.
   (i) Find a polynomial $g(x) \in \mathbb{Z}[x]$ that has the same Galois group as $f(x)$.
   (ii) Find the Galois group of $g(x)$.

5. Let $\alpha = \sqrt{7 + 3\sqrt{5}}$. Find the degree of the extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, and find $(1+\alpha)^{-1}$ in the form $a + b\alpha + c\alpha^2 + \ldots$, where $a, b, c, \cdots \in \mathbb{Q}$.
6. Let R be a commutative ring with 1. Find the center of $M_n(R)$? Justify your answer.

7. Let N be a positive integer. Let x be an integer relatively prime to N, d relatively prime to $\varphi(N)$, and $dd' \cong 1 \mod \varphi(N)$. Show that $y \cong x^d \mod N$ implies that $x \cong y^{d'} \mod N$.

8. Prove that $(x-1)(x-2)\cdots(x-n) - 1$ is irreducible over $\mathbb{Z}$ for all $n \geq 1$.

9. Let R be a ring with 1, and let M be a left R-module. The set of torsion elements is denoted
$$Tor(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$
   (a) Prove that if R is an integral domain then $Tor(M)$ is a submodule of M.
   (b) Give an example of a ring R and an R-module M such that $Tor(M)$ is not a submodule.

10. Let V be a vector space of finite dimension over a field F. If φ is any linear transformation from V to V, prove there is an integer m such that $\ker \varphi^m \cap \mathrm{im}\varphi^m = \{\mathbf{0}\}$.