Algebra PhD Qualifying Exam

January 2016

Answer all questions. Partial credit will be given.

(1) (i) Let G be a group and let H_1, H_2 be proper subgroups of G. Show that $G \neq H_1 \cup H_2$.
(ii) Is it possible for a group to be the union of three proper subgroups? (Give a proof that it cannot, or an example showing that some group can.)

(2) Prove Sylow's theorem: Let p be a prime and let G be a group of finite order $p^a m$ where $\gcd(p, m) = 1$. Prove that G has a subgroup of order p^a.

(3) Let M/L and L/K be algebraic extensions of fields. Show that M/K is also an algebraic extension of fields.

(4) Find the Galois group of the splitting field over $\mathbb{Q}$ of $x^8 - 2$.

(5) Let p be a prime and let $\mathcal{C}_n$ be the cyclic group of order n. Find (i) $|\mathrm{Aut}(\mathcal{C}_p \times \mathcal{C}_p)|$; (ii) $|\mathrm{Aut}(\mathcal{C}_2 \times \mathcal{C}_{p^2})|$.

(6) Let A be a commutative ring with 1. Let $N = \{x \mid x^n = 0 \text{ for some } n > 0\}$ be the set of nilpotent elements of A.
(a) Show that N is an ideal.
(b) Show that A/N contains no nonzero nilpotents.
(c) Find N when $A = \mathbb{Z}/90\mathbb{Z}$.

(7) Let F be a field and let M be an $n \times n$ matrix over F that has finite order.
(a) If $F = \mathbb{C}$ show that M is diagonalizable.
(b) Is it true that M is always diagonalizable for any F?

(8) Which of the following are true? If true, give a proof. If false, give an example to demonstrate.
(a) $\mathbb{C}[x, y]$ is a PID.
(b) If R is a PID and I is a prime ideal of R, then R/I is a PID.

(9) Let R be a commutative ring with 1, and let M be an R-module. Then $\mathrm{Hom}_R(R, M)$ is an R-module: $(r \cdot f)(s) = r \cdot (f(s))$ (for $f \in \mathrm{Hom}_R(R, M) \cong M, r, s \in R$). Show that $\mathrm{Hom}_R(R, M) \cong M$ as R-modules.

(10) Show that $2 \otimes 1$ is zero in $\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$ but not in $2\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$. (Hint for the second assertion: Prove that the map $\varphi : 2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ given by $\varphi(a, b) = ab/2$ is bilinear. What is $\varphi(2, 1)$? Explain why this gives the result.)