Algebra PhD Qualifying Exam - Winter 2014

All rings have unity. Each problem is worth 20 points. Do as many as you can.

1. How many elements of order 5 are contained in a group of order 20? Justify your answer.

2. List all abelian groups of order 400, up to isomorphism. Each isomorphism class of groups should appear on your list exactly once. Describe each group in both invariant factor and elementary divisor form, and indicate which is which.

3. Prove there is no simple group G of order 36. (Hint: Can you find a set with 4 elements on which G acts nontrivially?)

4. Let $\zeta = \exp(2\pi i/3)$ (where $\exp z = e^z$), and let $\beta = \zeta\sqrt[3]{2}$. Let $K = \mathbb{Q}(\beta)$. Prove that -1 cannot be written as a sum of squares in K.

5. Recall that the discriminant D of a cubic polynomial $x^3 + px + q$, is given by $D = -4p^3 - 27q^2$.
   (a) Determine the Galois group of $f(x) = x^3 + x + 1$ over $\mathbb{Q}$.
   (b) Determine the Galois group of $f(x)$ over $\mathbb{F}_3$.

6. How many roots does the polynomial $x^3 - x^2 + x - 1$ have in the field $\mathbb{F}_{27}$, where $\mathbb{F}_{27}$ is the field with 27 elements? Justify your answer.

7. Factor each of the following polynomials in $\mathbb{Q}[x]$. Explain your methods.
   (a) $x^3 + 27x + 213$
   (b) $x^4 + 2x^3 + 3x^2 + 2x + 1$

8. Let p be a prime and let A be an $n \times n$ matrix with integer entries such that $A^p = I$, $A \neq I$. Prove that $n \geq p - 1$.

9. An element of a ring R is said to be nilpotent if some power of x is zero.
   (a) Prove that if x is nilpotent then $1 + x$ is a unit in R.
   (b) Prove that if R is commutative, the nilpotent elements form an ideal. Is this true if R is noncommutative?

10. Prove or disprove: there exists an integral domain with exactly 10 elements.