PH.D. QUALIFYING EXAM SPRING 2012 - ALGEBRA

Answer all the questions. In each case justify your answer. Here $\mathbb{F}_p$ is the finite field with p elements.

1. Let G be a group of order p^n, where p is a prime.
   (i) Show that the center of G is non-trivial.
   (ii) Show that every maximal subgroup of G is normal.
2. Let G be a group of order $p^n m$, where p is a prime and $\gcd(p, m) = 1$. Show that G has a subgroup of order p^n.
3. Let G be the group with presentation
$$G = \langle a, b | a^9, b^4, b^{-1}ab = a^5 \rangle.$$
   (i) Find the order of G.
   (ii) Find the center Z of G.
   (iii) Find G/Z.
4. (i) Show that every a vector space has a basis. (Do not assume that V is finite dimensional.)
   (ii) Let V be a vector space of finite dimension and let $T : V \to V$ be a linear transformation. Show that $V = K \oplus W$ where $K = \ker(T)$ and $W \cong Image(T)$.
5. Show that if a finite ring R with 1 admits an injective (ring) homomorphism from a field, then the number of elements of R must be a power of a prime number. Is R necessarily a field?
6. For a field K and $n \geq 1$ let $J_{n,K}$ denote the $n \times n$ matrix over K whose (i, j) entry is equal to $(-1)^{i+j} \in K$.
   (a) Find the Jordan form of $J_{3,\mathbb{F}_2}$;
   (b) Find the Jordan form of $J_{3,\mathbb{F}_3}$.
   (c) Find the Jordan form of $J_{3,\mathbb{Q}}$.
7. Let R be a PID. Show that every non-zero prime ideal is maximal.
8. Find the Galois group of the polynomial $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$.
9. Let F be a field and let $f(x) \in F[x]$. Show that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible over F.
10. An R-module M is said to be *irreducible* if $M \neq \{0\}$ and M has no R-submodules except $\{0\}$ and M. Let V be a finite-dimensional vector space over a field k, and let $T : V \to V$ be a linear transformation. Then T gives V the structure of a $k[x]$-module. Prove that V is irreducible as a $k[x]$-module if and only if the characteristic polynomial of T is irreducible in $k[x]$.