# PH. D. QUALIFYING EXAM WINTER 2011 - ALGEBRA

Answer all the questions. For each question give appropriate proofs.

1. Classify up to isomorphism the non-abelian groups of order 8.

2. (Sylow's Theorem) Show that for any finite group $G$ of order $n = p^k m, \gcd(p, m) = 1$, there is a subgroup of order $p^k$.

3. Prove that the alternating group $A_5$ is a simple group.

4. (Gauss's Lemma) Let $R$ be a Unique Factorization Domain with field of fractions $F$ and let $p(x) \in R[x]$. Show that if $p(x)$ is reducible in $F[x]$, then it is reducible in $R[x]$.

5. Let $R = \mathbb{C}S_3$ be the group ring over the symmetric group $S_3$. Find two commuting, orthogonal, non-constant idempotents in $R$. (Recall that an idempotent is an $e \in R$ such that $e^2 = e$.)

6. Let $F$ be a field and let $M$ be an invertible matrix having finite order, with coefficients in $F$.
(a) If $F = \mathbb{C}$ show that $M$ is diagonalizable (i.e. similar to a diagonal matrix).
(b) If $F \neq \mathbb{C}$ is $M$ necessarily diagonalizable?

7. Find the Galois group of the polynomial $x^8 - 2 \in \mathbb{Q}[x]$.

8. Let $K/F$ be a finite extension of fields. Show that $K = F(\theta)$ for some $\theta \in K$ if and only if there are only finitely many subfields of $K$ containing $F$.

9. Let $R = \mathbb{Q}[x, y]$. Let $I = (x), J = (x, y)$ be ideals in $R$.
(a) Are $I, J$ prime ideals?
(b) Are $I, J$ maximal ideals?

10. Show that $R$ is a Unique Factorization Domain if and only if $R[x]$ is a Unique Factorization Domain.