# Ph.D. QUALIFIER EXAMINATION: ANALYSIS

## January 2018

**Instructions**: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY.

**Some Notation**.

1. $\mathbb{R}^k$ – Euclidean $k$-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $\Im z$ is the imaginary part of $z$.
4. $\Re z$ is the real part of $z$.
5. $\mathcal{B}_X$ – the Borel $\sigma$-algebra in $X$
6. $(X, \mathcal{M}, \mu)$ – a measure space where $X$ is a set, $\mathcal{M}$ is a $\sigma$-algebra of subsets of $X$, and $\mu$ is a measure on $\mathcal{M}$
7. a.e.$[\mu]$ – almost every with respect to the measure $\mu$
8. $m$ – Lebesgue measure on $\mathbb{R}^k$
9. $\|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}$ – the $L^p$-norm of a $\mu$-measurable function $f\colon X \to \mathbb{C}$
10. $\|f\|_\infty$ – the essential supremum of $f$
11. $p, q$ – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
12. $L^p(\mu)$ – the space of $\mu$-measurable functions $f\colon X \to \mathbb{C}$ with $\|f\|_p < \infty$
13. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f\colon \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
14. $\|\Gamma\| = \sup\{\|\Gamma x\| \colon x \in X, \|x\| \le 1\}$ – operator norm of a linear transformation $\Gamma\colon X \to Y$ where $X$ and $Y$ are normed linear spaces
15. $|\lambda|$ – the total variation of a measure $\lambda$.
16. $\lambda \ll \mu$ – the measure $\lambda$ is absolutely continuous with respect to the measure $\mu$
17. $\lambda \perp \mu$ – the measures $\lambda$ and $\mu$ are mutually singular
18. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of $\lambda$ with respect to $\mu$ where $\lambda \ll \mu$
19. $\operatorname{Lip}\alpha$ – the space of complex functions $f$ on $[a, b]$ for which $\sup_{x \ne y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \le 1$
20. $f * g$ – the convolution of $f$ and $g$: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y) \, dm(y)$
21. $C_c(X)$ – the continuous complex functions on $X$ with compact support
22. $C_0(X)$ – the continuous complex functions on a LCH space $X$ which vanish at infinity
23. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dm(x)$ – the Fourier transform.

# Questions

1. Compute the Fourier transform of $e^{-\pi x^2}$.

2. Compute $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2}\,dx$.

3. Assume $f(z)$ is analytic and bounded by one for $|z| \leq 1$.
   Prove that $\dfrac{|f'(z)|}{1-|f(z)|^2} \leq \dfrac{1}{1-|z|^2}$.

4. Assume $f(z)$ is analytic in a connected open set $\Omega$ and $a \in \Omega$. If $f^{(n)}(a) = 0$ for $n = 0, 1, 2, \cdots$, prove $f(z) = 0$ for all $z \in \Omega$.

5. Prove that a nonconstant analytic function maps open sets onto open sets.

6. State and prove Fatou's lemma (you may use the Monotone Convergence Theorem but you must state it clearly).

7. Define what a Hamel basis is and what a Schauder basis is. Is every Hamel basis a Schauder basis? Is every Schauder basis a Hamel basis? Explain.

8. Let $(X, A, \mu)$ be a measure space. Prove that $\mu$ is $\sigma$-finite if and only if there is a $\mu$-integrable function $f : X \to (0, \infty)$.

9. State Fubini's Theorem. Prove that there is a real-valued function $f(x, y)$ defined on $[0, 1] \times [0, 1]$, with Lebesgue measure on each $[0, 1]$, for which both iterated integrals of $f$ exist but are not equal.

10. Prove that if a normed space $X$ is finite dimensional, then every linear operator on $X$ is bounded. (You may find this fact useful. Let $\{x_1, x_2, \cdots, x_n\}$ be a linearly independent set of vectors in a normed space $X$. Then there is a number $c > 0$ such that for every choice of scalars $\alpha_1, \cdots, \alpha_n$, we have $\|\alpha_1 x_i + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|)$.)