Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Fall 2016

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10. REMEMBER TO WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $\Im z$ is the imaginary part of z.
4. $\mathcal{B}_X$ – the Borel σ-algebra in X
5. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
6. a.e.$[\mu]$ – almost every with respect to the measure μ
7. m – Lebesgue measure on $\mathbb{R}^k$
8. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f\colon X \to \mathbb{C}$
9. $\|f\|_\infty$ – the essential supremum of f
10. p, q – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
11. $L^p(\mu)$ – the space of μ-measurable functions $f\colon X \to \mathbb{C}$ with $\|f\|_p < \infty$
12. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f\colon \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
13. $\|\Gamma\| = \sup\{\|\Gamma x\| \colon x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma\colon X \to Y$ where X and Y are normed linear spaces
14. $|\lambda|$ – the total variation of a measure λ.
15. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
16. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
17. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
18. $\operatorname{Lip} \alpha$ – the space of complex functions f on $[a, b]$ for which $\sup\limits_{x \neq y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
19. $f * g$ – the convolution of f and g: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y) \, dm(y)$
20. $C_c(X)$ – the continuous complex functions on X with compact support
21. $C_0(X)$ – the continuous complex functions on a LCH space X which vanish at infinity
22. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dm(x)$ – the Fourier transform.

Questions

1. Suppose μ and λ are measures on a σ-algebra $\mathfrak{M}$, μ is positive, and λ is complex. Prove that the following two conditions are equivalent: a) λ is absolutely continuous with respect to μ. b) For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\lambda(E)| < \epsilon$ for all $E \in \mathfrak{M}$ with $\mu(E) < \delta$.

2. Let $f(z)$ be an entire function, and let $I(r) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$. Prove that $\ln I(r)$ is a convex function of $\ln r$.

3. Prove that every bounded positive linear operator on a complex Hilbert space is self-adjoint.

4. Compute $\displaystyle\int_{|z|=r} \frac{|dz|}{|z-a|^2}$, where $|a| \neq r$.

5. Assume that $\mu(X) < \infty$ and that $\{f_n\}$ is a sequence of complex measurable functions which converges pointwise at every point of X. Prove that for every $\epsilon > 0$ there exists a measurable set $E \subset X$ with $\mu(X - E) < \epsilon$ such that $\{f_n\}$ converges uniformly on E.

6. Show that a nonconstant analytic function maps open sets onto open sets.

7. Let $\{E_k\}$ be a sequence of measurable sets in X such that $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Show that almost all $x \in X$ lie in at most finitely many of the sets E_k.

8. Prove that every one-to-one conformal mapping of the unit disk onto itself is a linear transformation.

9. If A is an orthogonal projection on a Hilbert space $\mathcal{H}$ and if B is a positive linear operator on $\mathcal{H}$, prove that all eigenvalues of AB are nonnegative.

10. If $f(z)$ is analytic and $\Im f(z) \geq 0$ for $\Im z > 0$, show that

$$\left|\frac{f(z) - f(z_0)}{f(z) - \bar{f}(z_0)}\right| \leq \left|\frac{z - z_0}{z - \bar{z}_0}\right|.$$